[Note]
Measuring Lensing Effect on CMB with Minkowski Functionals

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1 Introduction

Several studies have explored the usefulness of Minkowski functionals (MFs) as a test for primordial non-Gaussianity (e.g., [1, 2]).

We adopt the cosmological parameters assuming a flat Lambda-CDM model consistent with the results obtained from Ref. [3]; the density parameter of baryon $\Omega_{\rm b}h^2 = 0.022$, of matter $\Omega_{\rm m}h^2 = 0.12$, dark energy density $\Omega_\Lambda = 0.69$, scalar spectral index $n_{\rm s} = 0.96$, scalar amplitude $A_{\rm s} = 2.2\times 10^{-9}$ and the optical depth, $\tau = 0.089$.

2 Lensing induced Minkowski functionals

2.1 Definition of Minkowski functionals

Given observed anisotropies X, a filtered map $\overline{X}$ is given by

$$\overline{X}(\hat{\boldsymbol{n}}) = \sum_{\ell m} f_\ell X_{\ell m} Y_{\ell m}(\hat{\boldsymbol{n}}) \,, \tag{1}$$

where f is a filtering function, and corresponds to a convolution kernel in real space.

MFs of the two-dimensional filtered map $\overline{X}$ is given as a function of threshold ν (see e.g.,[4]) :

$$V_0(\nu) = \int {\rm d}a \; \Xi(\overline{X}-\nu) \,, \tag{2}$$

$$V_1(\nu) = \frac{1}{4}\int {\rm d}a \; |\boldsymbol{\nabla}\overline{X}|\delta(\overline{X}-\nu) \,, \tag{3}$$

$$V_2(\nu) = \frac{1}{2\pi}\int {\rm d}a \; |\boldsymbol{\nabla}\overline{X}|\delta(\overline{X}-\nu)K \,, \tag{4}$$

where Ξ and K denotes the step function and principal curvature, respectively.

2.2 Analytical expression for Minkowski functionals in weak non-Gaussian regime

If the non-Gaussianity is weak, expectation values of 2D MFs becomes [2]

$$V_k(\nu) = A_k e^{-\nu^2/2} v_k(\nu) \,, \tag{5}$$

where

$$A_k \equiv \frac{\Gamma(2-k/2)\Gamma(k/2)q^k}{(2'i)^{(k+1)/2}} = \begin{cases} \frac{1}{\sqrt{2\pi}} & (k=0) \\ \frac{q}{8} & (k=1) \\ \frac{q^2}{(2\pi)^{3/2}} & (k=2) \end{cases} \; ; \qquad q = \frac{\sigma_1}{\sqrt{2}\,\sigma_0} \,, \tag{6}$$

$$\sigma_0^2 = \langle \overline{X}^2(\hat{\boldsymbol{n}})\rangle = \sum_\ell f_\ell^2 C_\ell \sum_m Y_{\ell,m}(\hat{\boldsymbol{n}})Y^*_{\ell,m}(\hat{\boldsymbol{n}}) = \frac{1}{4\pi}\sum_\ell (2\ell+1) f_\ell^2 C_\ell \,, \tag{7}$$

$$\sigma_1^2 = \langle X_{,i}(\hat{\boldsymbol{n}})X^{,i}(\hat{\boldsymbol{n}})\rangle = \frac{1}{4\pi}\sum_\ell (2\ell+1)\bar{\ell} f_\ell^2 C_\ell \,, \tag{8}$$

with $\bar{\ell} = \ell(\ell+1)$. We assume that f_ℓ is a Gaussian kernel multiplied by the top-hat function in harmonic space:

$$f_\ell = \begin{cases} \exp\left[-\bar{\ell}\frac{\theta_s^2}{2}\right] & (\ell_{\min} \le \ell \le \ell_{\max}) \\ 0 & (otherwise) \end{cases} . \tag{9}$$

The function $v_k(\nu)$ is expanded in terms of σ_0 as [2, 5]

$$v_k(\nu) = v_k^{(0)}(\nu) + v_k^{(1)}(\nu)\sigma_0 + v_k^{(2)}(\nu)\sigma_0^2 + \cdots , \tag{10}$$

with

$$v_k^{(0)}(\nu) = h_{k-1}(\nu) , \tag{11}$$

$$v_k^{(1)}(\nu) = \frac{S}{6}h_{k+2}(\nu) - \frac{kS_{\rm I}}{4}h_k(\nu) - \frac{k(k-1)S_{\rm II}}{4}h_{k-2}(\nu) , \tag{12}$$

$$v_0^{(2)}(\nu) = \frac{S^2}{72}h_5(\nu) + \frac{K}{24}h_3(\nu) . \tag{13}$$

$$v_1^{(2)}(\nu) = \frac{S^2}{72}h_6(\nu) + \frac{K - SS_{\rm I}}{24}h_4(\nu) - \frac{1}{12}\left(K_{\rm I} + \frac{3}{8}S_{\rm I}^2\right)h_2(\nu) - \frac{K_{\rm III}}{8} . \tag{14}$$

$$v_2^{(2)}(\nu) = \frac{S^2}{72}h_7(\nu) + \frac{K - 2SS_{\rm I}}{24}h_5(\nu) - \frac{1}{6}\left(K_{\rm I} + \frac{SS_{\rm II}}{2}\right)h_3(\nu) - \frac{1}{2}\left(K_{\rm II} + \frac{S_{\rm I}S_{\rm II}}{2}\right)h_1(\nu) . \tag{15}$$

The “probabilists” Hermite polynomials $h_k(\nu)$ are defined as

$$h_n(x) = e^{x^2/2}\left(-\frac{\mathrm{d}}{\mathrm{d}x}\right)^n e^{x^2/2} \tag{16}$$

$$h_{-1}(x) = e^{x^2/2}\int_x^\infty dt e^{t^2/2} = \sqrt{\frac{'i}{2}}\, e^{x^2/2}\mathrm{erfc}\left(\frac{x}{\sqrt{2}}\right) . \tag{17}$$

In particular,

$$\begin{aligned} &h_0(x) = 1 , && h_1(x) = x , && h_2(x) = x^2 - 1 , \\ &h_3(x) = x^3 - 3x , && h_4(x) = x^4 - 6x^2 + 3 , && h_5(x) = x^5 - 10x^3 + 15x , \\ &h_6(x) = x^6 - 15x^4 + 45x^2 - 15 , && h_7(x) = x^7 - 21x^5 + 105x^3 - 105x . \end{aligned} \tag{18}$$

2.3 Expression in terms of polyspectra

The bispectrum $B_{\ell_1\ell_2\ell_3}$ and trispectrum $T^{\ell_1\ell_2}_{\ell_3\ell_4}(L)$ are defined as [6, 7]

$$\langle \overline{X}_{\ell_1 m_1}\overline{X}_{\ell_2 m_2}\overline{X}_{\ell_3 m_3}\rangle_{\rm c} = \begin{pmatrix} \ell_1 & \ell_2 & \ell_3 \\ m_1 & m_2 & m_3 \end{pmatrix} B_{\ell_1\ell_2\ell_3} , \tag{19}$$

$$\langle \overline{X}_{\ell_1 m_1}\overline{X}_{\ell_2 m_2}\overline{X}_{\ell_3 m_3}\overline{X}_{\ell_4 m_4}\rangle_{\rm c} = \sum_{LM} \begin{pmatrix} \ell_1 & \ell_2 & L \\ m_1 & m_2 & -M \end{pmatrix}\begin{pmatrix} \ell_3 & \ell_4 & L \\ m_3 & m_4 & M \end{pmatrix}(-1)^M T^{\ell_1\ell_2}_{\ell_3\ell_4}(L) . \tag{20}$$

The reduced bispectrum is defined as

$$B_{\ell_1\ell_2\ell_3} = I_{\ell_1\ell_2\ell_3} b_{\ell_1\ell_2\ell_3} , \tag{21}$$

where the quantity $I_{\ell_1\ell_2\ell_3}$ is expressed in terms of the Wigner-3j symbols as

$$I_{\ell_1\ell_2\ell_3} = \sqrt{\frac{(2\ell_1+1)(2\ell_2+1)(2\ell_3+1)}{4\pi}}\begin{pmatrix} \ell_1 & \ell_2 & \ell_3 \\ 0 & 0 & 0 \end{pmatrix} . \tag{22}$$

The skewness parameters $S_{\rm A}$ are related to the reduced bispectrum as [2]

$$S_{\rm A} = \frac{3}{2\pi\sigma_0^4}\sum_{2\le\ell_1\le\ell_2\le\ell_3} f_{\ell_1}f_{\ell_2}f_{\ell_3}I^2_{\ell_1\ell_2\ell_3}s_{\rm A}^{\ell_1\ell_2\ell_3}b_{\ell_1\ell_2\ell_3} . \tag{23}$$

where

$$s^{\ell_1\ell_2\ell_3} = 1\,, \tag{24}$$

$$s_{\rm I}^{\ell_1\ell_2\ell_3} = -\frac{\overline{\ell}_1+\overline{\ell}_2+\overline{\ell}_3}{6q^2}\,, \tag{25}$$

$$s_{\rm II}^{\ell_1\ell_2\ell_3} = \frac{\overline{\ell}_1^2+\overline{\ell}_2^2+\overline{\ell}_3^2-2\overline{\ell}_1\overline{\ell}_2-2\overline{\ell}_2\overline{\ell}_3-2\overline{\ell}_3\overline{\ell}_1}{12q^2}\,. \tag{26}$$

Similarly, the Kurtosis parameters are given by [2]

$$K_{\rm A} = \frac{1}{4\pi\sigma_0^6}\sum_{\ell_1\ell_2\ell_3\ell_4 L}\frac{I_{\ell_1\ell_2 L}I_{\ell_3\ell_4 L}}{2L+1}\kappa_{\rm A}^{\ell_1\ell_2\ell_3\ell_4 L}f_{\ell_1}f_{\ell_2}f_{\ell_3}f_{\ell_4}T^{\ell_1\ell_2}_{\ell_3\ell_4}(L)\,, \tag{27}$$

where

$$\kappa = 1\,, \tag{28}$$

$$\kappa_{\rm I} = -\frac{\overline{\ell}_1+\overline{\ell}_2+\overline{\ell}_3+\overline{\ell}_4}{8q^2}\,, \tag{29}$$

$$\kappa_{\rm II} = \frac{\overline{L}^2-(\overline{\ell}_1+\overline{\ell}_2)(\overline{\ell}_3+\overline{\ell}_4)}{16q^4}\,, \tag{30}$$

$$\kappa_{\rm III} = \frac{(\overline{\ell}_1+\overline{\ell}_2-\overline{L})(\overline{\ell}_3+\overline{\ell}_4-\overline{L})}{32q^4}\,. \tag{31}$$

3 Lensing induced Minkowski functionals: Analytic expression

3.1 Lensing propagator

Let us first consider the lensed CMB temperature anisotropies. At the first order of lensing potential, the lensed temperature is given by [**?**]:

$$\tilde{\Theta}_{LM} = \Theta_{LM} + \sum_{\ell m}\sum_{\ell' m'}\Theta_{\ell' m'}(-1)^M\begin{pmatrix} L & \ell & \ell' \\ -M & m & m' \end{pmatrix} I_{L\ell\ell'}\frac{-\overline{L}+\overline{\ell}+\overline{\ell}'}{2}\phi_{\ell m}\,. \tag{32}$$

To simplify the following calculations, we introduce a function $f^{(\Theta\Theta)}$ which characterize a propagator in the viewpoint of mode-coupling theory:

$$\langle\tilde{\Theta}_{LM}\tilde{\Theta}_{L'M'}\rangle_{\rm CMB} = \sum_{\ell m}\begin{pmatrix} L & L' & \ell \\ M & M' & m \end{pmatrix} f^{(\Theta\Theta)}_{L\ell L'}\phi^*_{\ell m}\,, \tag{33}$$

where

$$f^{(\Theta\Theta)}_{L\ell L'} = \frac{-\overline{L}+\overline{\ell}+\overline{L}'}{2}I_{L\ell L'}C^{\Theta\Theta}_{L'} + (L\leftrightarrow L')\,. \tag{34}$$

In the case including polarization, we define

$$\langle\tilde{X}_{LM}\tilde{Y}_{L'M'}\rangle_{\rm CMB} = \sum_{\ell m}\begin{pmatrix} L & L' & \ell \\ M & M' & m \end{pmatrix} f^{(XY)}_{L\ell L'}\phi^*_{\ell m}\,. \tag{35}$$

3.2 Lensing induced Bispectrum

3.2.1 Temperature

From Eq. (33), we obtain at the first order of lensing potential as [6, **?**, **?**]

$$\begin{aligned}\langle\tilde{\Theta}_{\ell_1 m_1}\tilde{\Theta}_{\ell_2 m_2}\tilde{\Theta}_{\ell_3 m_3}\rangle_{\rm c} &= \langle\langle\tilde{\Theta}_{\ell_1 m_1}\tilde{\Theta}_{\ell_2 m_2}\rangle_{\rm CMB}\tilde{\Theta}_{\ell_3 m_3}\rangle + 2\text{ perms.} \\ &= \sum_{\ell m}\begin{pmatrix} \ell_1 & \ell_2 & \ell \\ m_1 & m_2 & m \end{pmatrix} f^{(\Theta\Theta)}_{\ell_1\ell\ell_2}\langle\phi^*_{\ell m}\widetilde{\Theta}_{\ell_3 m_3}\rangle + 2\text{ perms.} \\ &= \begin{pmatrix} \ell_1 & \ell_2 & \ell_3 \\ m_1 & m_2 & m_3 \end{pmatrix} f^{(\Theta\Theta)}_{\ell_1\ell_3\ell_2}C^{\phi\Theta}_{\ell_3} + 2\text{ perms.}\,.\end{aligned} \tag{36}$$

The bispectrum is then given by

$$\begin{aligned} B_{\ell_1\ell_2\ell_3} &= f^{(\Theta\Theta)}_{\ell_1\ell_3\ell_2} C^{\Theta\phi}_{\ell_3} + 2 \text{ perms.} \\ &= \frac{-\overline{\ell}_1 + \overline{\ell}_2 + \overline{\ell}_3}{2} I_{\ell_1\ell_2\ell_3} C^{\Theta\phi}_{\ell_2} \tilde{C}^{\Theta\Theta}_{\ell_3} + (5\,\text{perm.}) \\ &= I_{\ell_1\ell_2\ell_3}\left[\frac{-\overline{\ell}_1 + \overline{\ell}_2 + \overline{\ell}_3}{2} C^{\Theta\phi}_{\ell_2} \tilde{C}^{\Theta\Theta}_{\ell_3} + (5\,\text{perm.})\right] . \end{aligned} \tag{37}$$

Substituting (**??**) into the above equation, and with the symmetric property of $I_{\ell_1\ell_2\ell_3}$, we obtain the expression of the reduced bispectrum as

$$b_{\ell_1\ell_2\ell_3} = \frac{-\overline{\ell}_1 + \overline{\ell}_2 + \overline{\ell}_3}{2} C^{\Theta\phi}_{\ell_2} C^{\Theta\Theta}_{\ell_3} + (5\,\text{perm.})\,. \tag{38}$$

3.2.2 Polarization

There are a contribution from $C^{E\phi}_\ell$, but this contribution is quite small, and we neglect this term.

3.3 Lensing induced Bispectrum

3.3.1 Temperature

The lensing induced trispectrum is given by [7]

$$\begin{aligned} T_{\ell_1\ell_2\ell_3\ell_4}(L) &= C^{\phi\phi}_L f^{(\Theta\Theta)}_{\ell_2 L\ell_1} f^{(\Theta\Theta)}_{\ell_4 L\ell_3} + 2 \text{ perm.} \\ &= C^{\phi\phi}_L \left[\tilde{C}^{\Theta\Theta}_{\ell_2} \frac{\overline{L} + \overline{\ell}_2 - \overline{\ell}_1}{2} + (\ell_1 \leftrightarrow \ell_2)\right]\left[\tilde{C}^{\Theta\Theta}_{\ell_3} \frac{\overline{L} + \overline{\ell}_4 - \overline{\ell}_3}{2} + (\ell_3 \leftrightarrow \ell_4)\right] I_{\ell_1\ell_2 L} I_{\ell_3\ell_4 L} + 2 \text{ perm.}\,. \end{aligned} \tag{39}$$

The Kurtosis parameters are rewritten as

$$\begin{aligned} K_{\rm A} &= \frac{1}{4\pi\sigma_0^6} \sum_{\ell_1,\ell_2,\ell_3,\ell_4,L} \frac{I^2_{\ell_1\ell_2 L} I_{\ell_3\ell_4 L}}{2L+1} \kappa_{\rm A}^{\ell_1\ell_2\ell_3\ell_4 L} W_{\ell_1} W_{\ell_2} W_{\ell_3} W_{\ell_4} C^{\phi\phi}_L f^{(\Theta\Theta)}_{L\ell_1\ell_2} f_{L\ell_3\ell_4} \\ &= \frac{1}{4\pi\sigma_0^6} \sum_L \frac{C^{\phi\phi}_L}{2L+1} \sum_{\ell_1\ell_2\ell_3\ell_4} \kappa_{\rm A}^{\ell_1\ell_2\ell_3\ell_4 L} F_{L\ell_1\ell_2} F_{L\ell_3\ell_4}\,, \end{aligned} \tag{40}$$

where

$$F_{L\ell_1\ell_2} = I_{\ell_1\ell_2 L} W_{\ell_1} W_{\ell_2} f_{L\ell_1\ell_2}\,. \tag{41}$$

For $i = 0$ and 3, the sum in the Kurtosis parameter is reduced to

$$K_{(0)} = \frac{1}{4\pi\sigma_0^6} \sum_L \frac{C^{\phi\phi}_L}{2L+1} \left[\sum_{\ell_1,\ell_2} F_{L,\ell_1,\ell_2}\right]^2 , \tag{42}$$

$$K_{(3)} = \frac{1}{4\pi\sigma_0^6} \sum_L \frac{C^{\phi\phi}_L}{32q^4(2L+1)} \left[\sum_{\ell_1,\ell_2} (\{\ell_1\} + \{\ell_2\} - \{L\}) F_{L,\ell_1,\ell_2}\right]^2 . \tag{43}$$

For $i = 1$, the sum is decomposed into

$$\begin{aligned} K_{(1)} &= -\frac{1}{4\pi\sigma_0^6} \sum_L \frac{C^{\phi\phi}_L}{2L+1} \sum_{\ell_1,\ell_2,\ell_3,\ell_4} \frac{\{\ell_1\} + \{\ell_2\} + \{\ell_3\} + \{\ell_4\}}{8q^2} F_{L,\ell_1,\ell_2} F_{L,\ell_3,\ell_4} \\ &= -\frac{1}{4\pi\sigma_0^6} \sum_L \frac{C^{\phi\phi}_L}{8q^2(2L+1)} \sum_{\ell_1,\ell_2} \sum_{\ell_3,\ell_4} [\{\ell_1\} + \{\ell_2\} + \{\ell_3\} + \{\ell_4\}] F_{L,\ell_1,\ell_2} F_{L,\ell_3,\ell_4} \\ &= -\frac{1}{4\pi\sigma_0^6} \sum_L \frac{C^{\phi\phi}_L}{4q^2(2L+1)} \left(\sum_{\ell_1,\ell_2} F_{L,\ell_1,\ell_2}\right)\left(\sum_{\ell_1,\ell_2} [\{\ell_1\} + \{\ell_2\}] F_{L,\ell_1,\ell_2}\right) . \end{aligned} \tag{44}$$

Finally, for $i=2$, we obtain

$$
\begin{aligned}
K_{(2)} &= \frac{1}{4\pi\sigma_0^6}\sum_L \frac{C_L^{\phi\phi}}{2L+1}\sum_{\ell_1,\ell_2,\ell_3,\ell_4}\frac{\{L\}^2-(\{\ell_1\}+\{\ell_2\})(\{\ell_3\}+\{\ell_4\})}{16q^4}F_{L,\ell_1,\ell_2}F_{L,\ell_3,\ell_4} \\
&= \frac{1}{4\pi\sigma_0^6}\sum_L \frac{C_L^{\phi\phi}}{16q^4(2L+1)}\left\{\left(\{L\}\sum_{\ell_1,\ell_2}F_{L,\ell_1,\ell_2}\right)^2-\left(\sum_{\ell_1,\ell_2}(\{\ell_1\}+\{\ell_2\})F_{L,\ell_1,\ell_2}\right)^2\right\} .
\end{aligned}
\tag{45}
$$

Based on the above facts, we rewrite the Kurtosis parameters in the following forms:

$$
K_{(0)} = \frac{1}{4\pi\sigma_0^6}\sum_L \frac{C_L^{\phi\phi}}{2L+1}[\mathcal{A}_L^{(a)}]^2 , \tag{46}
$$

$$
K_{(1)} = -\frac{1}{4\pi\sigma_0^6}\sum_L \frac{C_L^{\phi\phi}}{4q^2(2L+1)}\mathcal{A}_L^{(a)}\mathcal{A}_L^{(b)} , \tag{47}
$$

$$
K_{(2)} = \frac{1}{4\pi\sigma_0^6}\sum_L \frac{C_L^{\phi\phi}}{16q^4(2L+1)}\left\{(\mathcal{A}_L^{(c)})^2-(\mathcal{A}_L^{(b)})^2\right\} , \tag{48}
$$

$$
K_{(3)} = \frac{1}{4\pi\sigma_0^6}\sum_L \frac{C_L^{\phi\phi}}{32q^4(2L+1)}\left[\mathcal{A}_L^{(b)}-\{L\}\mathcal{A}_L^{(a)}\right]^2 , \tag{49}
$$

where we define

$$
\mathcal{A}_L^{(a)} = \sum_{\ell_1,\ell_2}F_{L,\ell_1,\ell_2} , \tag{50}
$$

$$
\mathcal{A}_L^{(b)} = \sum_{\ell_1,\ell_2}[\{\ell_1\}+\{\ell_2\}]F_{L,\ell_1,\ell_2} , \tag{51}
$$

$$
\mathcal{A}_L^{(c)} = \sum_{\ell_1,\ell_2}\{L\}F_{L,\ell_1,\ell_2} . \tag{52}
$$

Note that F_{L,ℓ_1,ℓ_2} is symmetric under $\ell_1 \leftrightarrow \ell_2$, and the sum can be reduced to $\ell_1 \leq \ell_2$.

3.3.2 Polarization

A Useful Formulas

Following Ref.[8] and Ref. [**?**], we summarize formulas.

A.1 Wigner-$3j$ symbols

A.1.1 Symmetric properties of Wigner-$3j$ symbols

$$
\begin{pmatrix}\ell_1 & \ell_2 & \ell_3\\ m_1 & m_2 & m_3\end{pmatrix} = \begin{pmatrix}\ell_2 & \ell_3 & \ell_1\\ m_2 & m_3 & m_1\end{pmatrix} = \begin{pmatrix}\ell_3 & \ell_1 & \ell_2\\ m_3 & m_1 & m_2\end{pmatrix} = (-1)^{\ell_1+\ell_2+\ell_3}\begin{pmatrix}\ell_3 & \ell_1 & \ell_2\\ m_3 & m_1 & m_2\end{pmatrix} , \tag{53}
$$

$$
\begin{pmatrix}\ell_1 & \ell_2 & \ell_3\\ m_1 & m_2 & m_3\end{pmatrix} = (-1)^{\ell_1+\ell_2+\ell_3}\begin{pmatrix}\ell_3 & \ell_1 & \ell_2\\ -m_3 & -m_1 & -m_2\end{pmatrix} . \tag{54}
$$

A.1.2 Summation of Wigner-$3j$ symbols

$$
\sum_M (-1)^{L+M}\begin{pmatrix}\ell & L & L\\ -m & M & -M\end{pmatrix} = \delta_{\ell,0}\delta_{m,0}\sqrt{\frac{2L+1}{2\ell+1}} , \tag{55}
$$

$$
\sum_{M,M'}\begin{pmatrix}\ell & L & L'\\ -m & M & M'\end{pmatrix}\begin{pmatrix}\ell' & L & L'\\ -m' & M & M'\end{pmatrix} = \frac{1}{2\ell+1}\delta_{\ell,\ell'}\delta_{m,m'} . \tag{56}
$$

A.1.3 Recursion relation: L_1, L_2 and L_3

From Eq.(25) in Sec.8.6 of Ref.[8],

$$a \begin{pmatrix} L_1 & \ell & L_2 \\ M_1 & M_2 & M_3 \end{pmatrix} = b \begin{pmatrix} L_1 & \ell & L_2+1 \\ M_1 & M_2 & M_3 \end{pmatrix} + c \begin{pmatrix} L_1 & \ell & L_2+2 \\ M_1 & M_2 & M_3 \end{pmatrix} \tag{57}$$

where the coefficients in general case are described by

$$\begin{aligned} a &= \left\{(L_2+1)^2 - M_3^2\right\}^{1/2} \alpha, \\ b &= -M_2\beta_1 + M_3\beta_2, \\ c &= -\left\{(L_2+2) - M_3^2\right\}^{1/2} \gamma \end{aligned} \tag{58}$$

The quantities α, β_i, γ are

$$\begin{aligned} \alpha &= (L_2+2)\sqrt{(-L_2+\ell+L_1)(L_2-\ell+L_1+1)(L_2+\ell-L_1+1)(L_2+\ell+L_1+2)}, \\ \beta_1 &= 2(L_2+1)(L_2+2)(2L_2+3) \\ \beta_2 &= (2L_2+3)[(L_2+1)(L_2+2)+\ell(\ell+1)-L_1(L_1-1)], \\ \gamma &= (L_2+1)\sqrt{(-L_2+\ell+L_1-1)(L_2-\ell+L_1+2)(L_2+\ell-L_1+2)(L_2+\ell+L_1+3)}, \end{aligned} \tag{59}$$

In particular,

- $(M_1 = 0, M_2 = 0)$,

$$a = (L_2+1)\alpha, \qquad b = 0, \qquad c = -(L_2+2)\gamma \tag{60}$$

- $(M_1 = 1, M_2 = -1)$,

$$a = (L_2+1)\alpha, \qquad b = \beta_1, \qquad c = -(L_2+2)\gamma \tag{61}$$

- $(M_1 = 2, M_2 = 0)$,

$$a = \left\{(L_2+1)^2 - 4\right\}^{1/2} \alpha, \qquad b = 2\beta_2, \qquad c = -\left\{(L_2+2)^2 - 4\right\}^{1/2} \gamma \tag{62}$$

- $(M_1 = 1, M_2 = 1)$,

$$a = \left\{(L_2+1)^2 - 4\right\}^{1/2} \alpha, \qquad b = -\beta_1 + 2\beta_2, \qquad c = -\left\{(L_2+2)^2 - 4\right\}^{1/2} \gamma \tag{63}$$

- $(M_1 = 2, M_2 = 1)$,

$$a = \left\{(L_2+1)^2 - 9\right\}^{1/2} \alpha, \qquad b = -\beta_1 + 3\beta_2, \qquad c = -\left\{(L_2+2)^2 - 9\right\}^{1/2} \gamma \tag{64}$$

A.1.4 Recursion relation: M_1, M_2 and M_3

From Eq.(4) in Sec.8.6 of Ref.[8],

$$\begin{aligned} -\sqrt{(L_3' m M_3)(L_3 \mp M_3 + 1)} \begin{pmatrix} L_1 & L_2 & L_3 \\ M_1 & M_2 & -M_3' m 1 \end{pmatrix} &= \sqrt{(L_1 \mp M_1)(L_1' m M_1 + 1)} \begin{pmatrix} L_1 & L_2 & L_3 \\ M_1' m 1 & M_2 & -M_3 \end{pmatrix} \\ &+ \sqrt{(L_2 \mp M_2)(L_2' m M_2 + 1)} \begin{pmatrix} L_1 & L_2 & L_3 \\ M_1 & M_2' m 1 & -M_3 \end{pmatrix}. \end{aligned} \tag{65}$$

In particular,

$$\begin{aligned} -\sqrt{(L_3+1)L_3} \begin{pmatrix} L_1 & L_2 & L_3 \\ 0 & 0 & 0 \end{pmatrix} &= \sqrt{L_1(L_1+1)} \begin{pmatrix} L_1 & L_2 & L_3 \\ 1 & 0 & -1 \end{pmatrix} \\ &+ \sqrt{L_2(L_2+1)} \begin{pmatrix} L_1 & L_2 & L_3 \\ 0 & 1 & -1 \end{pmatrix}. \end{aligned} \tag{66}$$

A.2 Hermite polynomials

A.2.1 Definition

The "probabilists" Hermite polynomials are defined as

$$h_n(x) = e^{x^2/2}\left(-\frac{\mathrm{d}}{\mathrm{d}x}\right)^n e^{x^2/2} \tag{67}$$

$$h_{-1}(x) = e^{x^2/2}\int_x^\infty dt e^{t^2/2} = \sqrt{\frac{\prime i}{2}}\, e^{x^2/2}\mathrm{erfc}\left(\frac{x}{\sqrt{2}}\right) . \tag{68}$$

There are another definition of Hermite polynomials which is given by

$$H_n(x) = e^{x^2}\left(-\frac{\mathrm{d}}{\mathrm{d}x}\right)^n e^{-x^2} = e^{x^2/2}\left(x-\frac{\mathrm{d}}{\mathrm{d}x}\right)^n e^{-x^2/2} . \tag{69}$$

These two polynomials are related to

$$H_n(x) = 2^{n/2} h_n(\sqrt{2}\,x) . \tag{70}$$

A.2.2 Properties

The Hermite polynomials are orthogonal with respect to the weight:

$$\int_{-\infty}^{\infty} dx h_n(x) h_m(x) e^{-x^2} = \sqrt{\prime i}\, n!\delta_{m,n} \tag{71}$$

$$\int_{-\infty}^{\infty} dx H_n(x) H_m(x) e^{-x^2} = \sqrt{2\prime i}\, 2^n n!\delta_{m,n} . \tag{72}$$

The recursion relation is

$$h_{n+1}(x) = x h_n(x) - \frac{\mathrm{d}h_n(x)}{\mathrm{d}x} , \qquad \frac{\mathrm{d}h_n(x)}{\mathrm{d}x} = n h_{n-1}(x) \tag{73}$$

$$H_{n+1}(x) = 2x H_n(x) - \frac{\mathrm{d}H_n(x)}{\mathrm{d}x} , \qquad \frac{\mathrm{d}H_n(x)}{\mathrm{d}x} = 2n H_{n-1}(x) . \tag{74}$$

The explicit expression for Hermite polynomials is

$$H_n(x) = \begin{cases} n!\displaystyle\sum_{m=0}^{n/2}\frac{(-1)^{n/2-m}}{(2m)!(n/2-m)!}(2x)^{2m} & (n=\mathrm{even}) \\ \\ n!\displaystyle\sum_{m=0}^{(n-1)/2}\frac{(-1)^{(n-1)/2-m}}{(2m+1)!((n-1)/2-m)!}(2x)^{2m+1} & (n=\mathrm{odd}) \end{cases} . \tag{75}$$

The generating function is

$$\exp(xt - t^2/2) = \sum_{n=0}^{\infty} h_n(t)\frac{t^n}{n!} , \qquad \exp(2xt - t^2) = \sum_{n=0}^{\infty} H_n(t)\frac{t^n}{n!} . \tag{76}$$

The Hermite polynomials are related to the Laguerre polynomials as

$$H_{2n}(x) = (-4)^n n! L_n^{-1/2}(x^2) = 4^n n!\sum_{i=0}^{n}(-1)^{n-i}\binom{n-1/2}{n-i}\frac{x^{2i}}{i!} , \tag{77}$$

$$H_{2n+1}(x) = 2(-4)^n n! x L_n^{-1/2}(x^2) = 2\times 4^n n!\sum_{i=0}^{n}(-1)^{n-i}\binom{n+1/2}{n-i}\frac{x^{2i+1}}{i!} . \tag{78}$$

B Calculation of Minkowski functionals

In this section,

B.1 Calculation of Wigner-$3j$ symbols

We first show our method of calculating MFs described in Eqs. (11) and (12). At the first step, we compute σ_0 and σ_1 from Eqs. (7) and (8). The angular power spectrum of lensed and unlensed CMB temperature is computed using CAMB [9]. The reduced bispectrum is computed according to Eq. (37). We assume the Gaussian smoothing kernel given in Eq. (9). To compute the skewness, we need

$$\begin{pmatrix} \ell_1 & \ell_2 & \ell_3 \\ 0 & 0 & 0 \end{pmatrix}, \tag{79}$$

under the condition, $\ell_1 \le \ell_2 \le \ell_3$. Denoting $\ell_1 + \ell_2 + \ell_3 = 2g$, this quantity is expressed as

$$\begin{pmatrix} \ell_1 & \ell_2 & \ell_3 \\ 0 & 0 & 0 \end{pmatrix} = \begin{cases} (-1)^g \left[\dfrac{(2g-2\ell_1)!(2g-2\ell_2)!(2g-2\ell_3)!}{(2g+1)!}\right]^{1/2} \dfrac{g!}{(g-\ell_1)!(g-\ell_2)!(g-\ell_3)!} & (2g = \text{even})\,, \\ 0 \quad (2g = \text{odd})\,. \end{cases} \tag{80}$$

But the above expression is not useful for calculating Wigner-$3j$ symbols, and we use the recursive formulas to compute the Wigner-$3j$ symbols as described in the followings.

B.1.1 Initial condition

Assuming $\ell_3 = \ell_2$ or $\ell_3 = \ell_2 + 1$ in Eq.(79), we obtain

$$\begin{aligned} \begin{pmatrix} \ell_1 & \ell_2+1 & \ell_2+1 \\ 0 & 0 & 0 \end{pmatrix} &= (-1)\left[\frac{(2g-2\ell_1+2)(2g-2\ell_1+1)}{(2g+3)(2g+2)}\right]^{1/2} \frac{g+1}{g+1-\ell_1} \begin{pmatrix} \ell_1 & \ell_2 & \ell_2 \\ 0 & 0 & 0 \end{pmatrix} \\ &= (-1)\left[\frac{(2\ell_2-\ell_1+2)(2\ell_2-\ell_1+1)}{(2\ell_2+\ell_1+3)(2\ell_2+\ell_1+2)}\right]^{1/2} \frac{2\ell_2+\ell_1+2}{2\ell_2-\ell_1+2} \begin{pmatrix} \ell_1 & \ell_2 & \ell_2 \\ 0 & 0 & 0 \end{pmatrix}, \\ \begin{pmatrix} \ell_1 & \ell_2+1 & \ell_2+2 \\ 0 & 0 & 0 \end{pmatrix} &= (-1)\left[\frac{(2g-2\ell_1+2)(2g-2\ell_1+1)}{(2g+3)(2g+2)}\right]^{1/2} \frac{g+1}{g+1-\ell_1} \begin{pmatrix} \ell_1 & \ell_2 & \ell_2+1 \\ 0 & 0 & 0 \end{pmatrix} \\ &= (-1)\left[\frac{(2\ell_2-\ell_1+3)(2\ell_2-\ell_1+2)}{(2\ell_2+\ell_1+4)(2\ell_2+\ell_1+3)}\right]^{1/2} \frac{2\ell_2+\ell_1+3}{2\ell_2-\ell_1+3} \begin{pmatrix} \ell_1 & \ell_2 & \ell_2+1 \\ 0 & 0 & 0 \end{pmatrix}. \end{aligned} \tag{81}$$

Further, for $\ell_1 = \ell_2 = \ell_3$ or $\ell_1 = \ell_2 = \ell_3 + 1$, we find

$$\begin{aligned} \begin{pmatrix} \ell_1 & \ell_1 & \ell_1 \\ 0 & 0 & 0 \end{pmatrix} &= (-1)^g \left[\frac{(\ell_1)!(\ell_1)!(\ell_1)!}{(3\ell_1+1)!}\right]^{1/2} \frac{(3\ell_1/2)!}{(\ell_1/2)!(\ell_1/2)!(\ell_1/2)!} \\ &= \frac{(-1)^{3\alpha}}{\sqrt{6\alpha+1}} \left[\left(\frac{(2\alpha)!}{(\alpha)!(\alpha)!}\right)^3 \frac{(3\alpha)!^2}{(6\alpha)!}\right]^{1/2} = \frac{(-1)^\alpha}{\sqrt{6\alpha+1}} \left[\prod_{i=0}^{\alpha-1} \frac{2\alpha-i}{\alpha-i}\right]^{3/2} \left[\prod_{i=0}^{3\alpha-1} \frac{\alpha-i/3}{2\alpha-i/3}\right]^{1/2}, \\ \begin{pmatrix} \ell_1 & \ell_1 & \ell_1+1 \\ 0 & 0 & 0 \end{pmatrix} &= (-1)^g \left[\frac{(\ell_1+1)!(\ell_1+1)!(\ell_1-1)!}{(3\ell_1+2)!}\right]^{1/2} \frac{((3\ell_1+1)/2)!}{(((\ell_1+1)/2)!)^2((\ell_1-1)/2)!} \\ &= (-1)^{\ell_1+\beta+1} \left[\frac{(2\beta+2)!(2\beta+2)!(2\beta)![(3\beta+2)!]^2}{[(\beta+1)!(\beta+1)!(\beta)!]^2(6\beta+5)!}\right]^{1/2} \\ &= \frac{2(2\beta+1)(3\beta+2)(3\beta+1)}{(\beta+1)\sqrt{(6\beta+2)(6\beta+3)(6\beta+4)(6\beta+5)}} \begin{pmatrix} 2\beta & 2\beta & 2\beta \\ 0 & 0 & 0 \end{pmatrix}, \end{aligned} \tag{82}$$

with $\ell_1 = 2\alpha$ and $\ell_1 - 1 = 2\beta$.

B.1.2 Recursion relation: ℓ_1, ℓ_2 and ℓ_3

From Eq.(25) in Sec.8.6 of Ref.[8],

$$\begin{pmatrix} \ell_1 & \ell_2 & \ell_3+2 \\ 0 & 0 & 0 \end{pmatrix} = \sqrt{\frac{(-\ell_3+\ell_2+\ell_1)(\ell_3-\ell_2+\ell_1+1)(\ell_3+\ell_2-\ell_1+1)(\ell_3+\ell_2+\ell_1+2)}{(-\ell_3+\ell_2+\ell_1-1)(\ell_3-\ell_2+\ell_1+2)(\ell_3+\ell_2-\ell_1+2)(\ell_3+\ell_2+\ell_1+3)}} \begin{pmatrix} \ell_1 & \ell_2 & \ell_3 \\ 0 & 0 & 0 \end{pmatrix}. \tag{83}$$

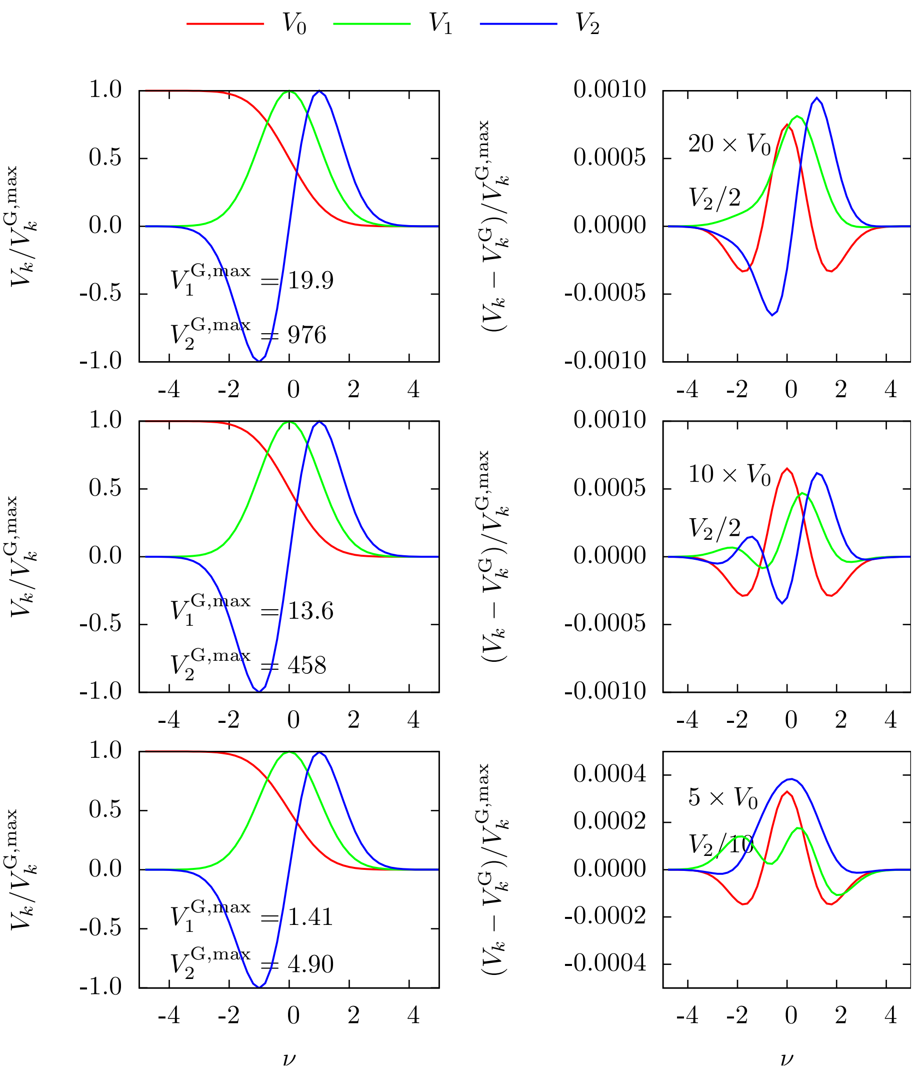

Figure 1: The lensing induced MFs for $(\theta_s, \ell_{\max}) = (5', 3000)$ (top), $(10', 1500)$ (middle) and $(100', 300)$ (bottom), assuming cosmic-variance limited experiment. V_k^{G} and $V_k^{\mathrm{G,max}}$ are the unlensed MFs and the maximum of $V_k^{\mathrm{G}}(\nu)$, respectively.

B.2 Results

In this section, we show the results of Minkowski functionals induced by CMB lensing. We assume a cosmic-variance limited (CV-limit) experiment up to the maximum multipole, $\ell_{\max} = 3000$, which approximately corresponds to the ACTPol/SPTpol like experiment. We consider three cases for the Gaussian smoothing radius, θ_{s}, and maximum multipole, $\ell_{\max}$; $(\theta_s, \ell_{\max}) = (5', 3000)$, $(10', 1500)$ and $(100', 300)$.

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