Lensing tools in flatsky

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Abstract

I will describe an explicit algorithm for computing the estimators, normalization and (diagonal) RDN0.

Contents

1 Introduction

Here, I will define the diagonal RDN0 given by [1].

1.1 Quadratic estimator

The distortion fields x described above induce the off-diagonal elements of the covariance ($\ell \neq \ell'$ or $m \neq m'$),

$$\langle \widetilde{X}_{\boldsymbol{L}} \widetilde{Y}_{\boldsymbol{\ell}-\boldsymbol{L}} \rangle_{\rm CMB} = f_{\boldsymbol{\ell},\boldsymbol{L}} x_{\boldsymbol{\ell}} \,, \tag{1}$$

where $\langle\cdots\rangle_{\rm CMB}$ denotes the ensemble average over the primary CMB anisotropies with a fixed realization of the distortion fields. With a quadratic combination of observed CMB anisotropies, $\hat{X}$ and $\hat{Y}$, the general quadratic estimators are formed as

$$x_{\boldsymbol{\ell}} = A_\ell^{x,{\rm XY}} \int \frac{{\rm d}^2\boldsymbol{L}}{(2\pi)^2} \frac{1}{\Delta^{\rm XY}} f^{x,XY}_{\boldsymbol{\ell},\boldsymbol{L}} F^X_L \hat{X}_{\boldsymbol{L}} F_{L'} \hat{Y}_{L'} \,. \tag{2}$$

Here, $x = \phi$, $A_\ell^{x,{\rm XY}}$ is the normalization and $\Delta^{\rm XX} = 2$, $\Delta^{\rm EB} = \Delta^{\rm TB} = 1$, and F_L is a diagonal filtering for the input multipoles. If we simply use the inverse variance, we may choose $F^X_L = 1/\hat{C}_L^{\rm XX,th}$. $\boldsymbol{L}' = \boldsymbol{\ell} - \boldsymbol{L}$.

1.2 Convolution

Let A and B be CMB fluctuations, satisfying $A^*_{\boldsymbol{\ell}} = A_{-\boldsymbol{\ell}}$ and $B^*_{\boldsymbol{\ell}} = B_{-\boldsymbol{\ell}}$. In general, we obtain

$$X_{\boldsymbol{\ell}} \equiv \int\frac{\mathrm{d}^2\boldsymbol{L}}{(2\pi)^2}\,[x_{\boldsymbol{L}}y_{\boldsymbol{L}'} \pm \text{c.c.}]A_{\boldsymbol{L}}B_{\boldsymbol{L}'} = \int\mathrm{d}^2\hat{\boldsymbol{n}}\;\mathrm{e}^{-\mathrm{i}\hat{\boldsymbol{n}}\cdot\boldsymbol{\ell}}\,[A(\hat{\boldsymbol{n}})B(\hat{\boldsymbol{n}}) \pm \text{c.c.}]\,, \tag{3}$$

where x and y are scalar and

$$A(\hat{\boldsymbol{n}}) \equiv \int\frac{\mathrm{d}^2\boldsymbol{L}}{(2\pi)^2}\,\mathrm{e}^{\mathrm{i}\boldsymbol{L}\cdot\hat{\boldsymbol{n}}}\,x_{\boldsymbol{L}}A_{\boldsymbol{L}}\,, \tag{4}$$

$$B(\hat{\boldsymbol{n}}) \equiv \int\frac{\mathrm{d}^2\boldsymbol{L}}{(2\pi)^2}\,\mathrm{e}^{\mathrm{i}\boldsymbol{L}\cdot\hat{\boldsymbol{n}}}\,y_{\boldsymbol{L}}B_{\boldsymbol{L}}\,. \tag{5}$$

Then we obtain

$$\int\frac{\mathrm{d}^2\boldsymbol{L}}{(2\pi)^2}\,\Re[x_{\boldsymbol{L}}y_{\boldsymbol{L}'}]A_{\boldsymbol{L}}B_{\boldsymbol{L}'} = \int\mathrm{d}^2\hat{\boldsymbol{n}}\;\mathrm{e}^{-\mathrm{i}\hat{\boldsymbol{n}}\cdot\boldsymbol{\ell}}\,\Re[A(\hat{\boldsymbol{n}})B(\hat{\boldsymbol{n}})]\,, \tag{6}$$

$$\int\frac{\mathrm{d}^2\boldsymbol{L}}{(2\pi)^2}\,\Im[x_{\boldsymbol{L}}y_{\boldsymbol{L}'}]A_{\boldsymbol{L}}B_{\boldsymbol{L}'} = \int\mathrm{d}^2\hat{\boldsymbol{n}}\;\mathrm{e}^{-\mathrm{i}\hat{\boldsymbol{n}}\cdot\boldsymbol{\ell}}\,\Im[A(\hat{\boldsymbol{n}})B(\hat{\boldsymbol{n}})]\,. \tag{7}$$

If either $x_{-\boldsymbol{L}} = -x_{\boldsymbol{L}}$ or $y_{-\boldsymbol{L}} = y_{\boldsymbol{L}}$, we find

$$\int\frac{\mathrm{d}^2\boldsymbol{L}}{(2\pi)^2}\,\Re[x_{\boldsymbol{L}}y_{\boldsymbol{L}'}]A_{\boldsymbol{L}}B_{\boldsymbol{L}'} = \mathrm{i}\int\mathrm{d}^2\hat{\boldsymbol{n}}\;\mathrm{e}^{-\mathrm{i}\hat{\boldsymbol{n}}\cdot\boldsymbol{\ell}}\,\Im[A(\hat{\boldsymbol{n}})B(\hat{\boldsymbol{n}})]\,, \tag{8}$$

$$\int\frac{\mathrm{d}^2\boldsymbol{L}}{(2\pi)^2}\,\Im[x_{\boldsymbol{L}}y_{\boldsymbol{L}'}]A_{\boldsymbol{L}}B_{\boldsymbol{L}'} = -\mathrm{i}\int\mathrm{d}^2\hat{\boldsymbol{n}}\;\mathrm{e}^{-\mathrm{i}\hat{\boldsymbol{n}}\cdot\boldsymbol{\ell}}\,\Re[A(\hat{\boldsymbol{n}})B(\hat{\boldsymbol{n}})]\,. \tag{9}$$

2 Lensing

For lensing, the weight functions are given as [2]

$$f^{\phi,\Theta\Theta}_{\boldsymbol{\ell},\boldsymbol{L}} = C^{\Theta\Theta}_L\boldsymbol{\ell}\cdot\boldsymbol{L} + C^{\Theta\Theta}_{L'}\boldsymbol{\ell}\cdot\boldsymbol{L}'\,, \tag{10}$$

$$f^{\phi,\Theta E}_{\boldsymbol{\ell},\boldsymbol{L}} = C^{\Theta\mathrm{E}}_L\boldsymbol{\ell}\cdot\boldsymbol{L}\cos 2\varphi_{L,L'} + C^{\Theta\mathrm{E}}_{L'}\boldsymbol{\ell}\cdot\boldsymbol{L}'\,, \tag{11}$$

$$f^{\phi,\Theta B}_{\boldsymbol{\ell},\boldsymbol{L}} = C^{\Theta\mathrm{E}}_L\boldsymbol{\ell}\cdot\boldsymbol{L}\sin 2\varphi_{L,L'}\,, \tag{12}$$

$$f^{\phi,EE}_{\boldsymbol{\ell},\boldsymbol{L}} = [\boldsymbol{\ell}\cdot\boldsymbol{L}C^{\mathrm{EE}}_L + \boldsymbol{\ell}\cdot\boldsymbol{L}'C^{\mathrm{EE}}_{L'}]\cos 2\varphi_{L,L'}\,, \tag{13}$$

$$f^{\phi,EB}_{\boldsymbol{\ell},\boldsymbol{L}} = [\boldsymbol{\ell}\cdot\boldsymbol{L}C^{\mathrm{EE}}_L - \boldsymbol{\ell}\cdot\boldsymbol{L}'C^{\mathrm{BB}}_{L'}]\sin 2\varphi_{L,L'}\,, \tag{14}$$

$$f^{\phi,BB}_{\boldsymbol{\ell},\boldsymbol{L}} = [\boldsymbol{\ell}\cdot\boldsymbol{L}C^{\mathrm{BB}}_L + \boldsymbol{\ell}\cdot\boldsymbol{L}'C^{\mathrm{BB}}_{L'}]\cos 2\varphi_{L,L'}\,, \tag{15}$$

where $\boldsymbol{L}' = \boldsymbol{\ell} - \boldsymbol{L}$.

2.1 Estimator

In the following, we define the inverse-variance filtered multipoles as

$$\overline{E}_{\boldsymbol{\ell}} = \frac{\widehat{E}_{\boldsymbol{\ell}}}{\hat{C}^{\mathrm{EE}}_\ell}\,, \qquad \overline{B}_{\boldsymbol{\ell}} = \frac{\widehat{B}_{\boldsymbol{\ell}}}{\hat{C}^{\mathrm{BB}}_\ell}\,. \tag{16}$$

and

$$\overline{X}^s(\hat{\boldsymbol{n}}) = \int\frac{\mathrm{d}^2\boldsymbol{L}}{(2\pi)^2}\,\mathrm{e}^{\mathrm{i}\boldsymbol{L}\cdot\hat{\boldsymbol{n}}}\,\overline{X}_{\boldsymbol{L}}e^{si\varphi_{\boldsymbol{L}}}\,, \tag{17}$$

$$\boldsymbol{\mathcal{X}}^s(\hat{\boldsymbol{n}}) = \int\frac{\mathrm{d}^2\boldsymbol{L}}{(2\pi)^2}\,\mathrm{e}^{\mathrm{i}\boldsymbol{L}\cdot\hat{\boldsymbol{n}}}\,\boldsymbol{L}\widetilde{C}^{XX}_L\overline{X}_{\boldsymbol{L}}e^{si\varphi_{\boldsymbol{L}}}\,. \tag{18}$$

The quadratic EE estimator is given by

$$\begin{aligned}
\bar{x}^{EE}_{\boldsymbol{\ell}} &= \boldsymbol{\ell}\odot_x \int\frac{\mathrm{d}^2\boldsymbol{L}}{(2\pi)^2}\,\frac{\boldsymbol{L}\widetilde{C}^{\rm EE}_L+\boldsymbol{L}'\widetilde{C}^{\rm EE}_{L'}}{2}\cos 2\varphi_{\boldsymbol{L}\boldsymbol{L}'}\overline{E}_{\boldsymbol{L}}\overline{E}_{\boldsymbol{L}'} \\
&= \boldsymbol{\ell}\odot_x \int\frac{\mathrm{d}^2\boldsymbol{L}}{(2\pi)^2}\,\Re\left[(\boldsymbol{L}\widetilde{C}^{\rm EE}_L+\boldsymbol{L}'\widetilde{C}^{\rm EE}_{L'})\frac{\mathrm{e}^{2\mathrm{i}\varphi_{\boldsymbol{L}}}\,\mathrm{e}^{-2\mathrm{i}\varphi_{\boldsymbol{L}'}}}{2}\right]\overline{E}_{\boldsymbol{L}}\overline{E}_{\boldsymbol{L}'} \\
&= \boldsymbol{\ell}\odot_x \int\frac{\mathrm{d}^2\boldsymbol{L}}{(2\pi)^2}\,\Re\left[\boldsymbol{L}\widetilde{C}^{\rm EE}_L\,\mathrm{e}^{2\mathrm{i}\varphi_{\boldsymbol{L}}}\,\mathrm{e}^{-2\mathrm{i}\varphi_{\boldsymbol{L}'}}\right]\overline{E}_{\boldsymbol{L}}\overline{E}_{\boldsymbol{L}'} \\
&= \boldsymbol{\ell}\odot_x \int \mathrm{d}^2\hat{\boldsymbol{n}}\;\mathrm{e}^{-\mathrm{i}\hat{\boldsymbol{n}}\cdot\boldsymbol{\ell}}\,\mathrm{i}\Im[\boldsymbol{\mathcal{E}}^2(\hat{\boldsymbol{n}})\overline{E}^{-2}(\hat{\boldsymbol{n}})]\,, \qquad (19)
\end{aligned}$$

The quadratic EB estimator becomes

$$\begin{aligned}
\bar{x}^{EB}_{\boldsymbol{\ell}} &= \boldsymbol{\ell}\odot_x \int\frac{\mathrm{d}^2\boldsymbol{L}}{(2\pi)^2}\,(\boldsymbol{L}\widetilde{C}^{\rm EE}_L+\boldsymbol{L}'\widetilde{C}^{\rm BB}_{L'})\sin 2\varphi_{\boldsymbol{L},\boldsymbol{L}'}\overline{E}_{\boldsymbol{L}}\overline{B}_{\boldsymbol{L}'} \\
&= \boldsymbol{\ell}\odot_x \int\frac{\mathrm{d}^2\boldsymbol{L}}{(2\pi)^2}\,\Im\left[(\boldsymbol{L}\widetilde{C}^{\rm EE}_L+\boldsymbol{L}'\widetilde{C}^{\rm BB}_{L'})\,\mathrm{e}^{2\mathrm{i}\varphi_{\boldsymbol{L}}}\,\mathrm{e}^{-2\mathrm{i}\varphi_{\boldsymbol{L}'}}\right]\overline{E}_{\boldsymbol{L}}\overline{B}_{\boldsymbol{L}'} \\
&= -\mathrm{i}\boldsymbol{\ell}\odot_x\left(\int \mathrm{d}^2\hat{\boldsymbol{n}}\;\mathrm{e}^{-\mathrm{i}\hat{\boldsymbol{n}}\cdot\boldsymbol{\ell}}\,\Re[\boldsymbol{\mathcal{E}}^2(\hat{\boldsymbol{n}})\overline{B}^{-2}(\hat{\boldsymbol{n}})-\boldsymbol{\mathcal{B}}^2(\hat{\boldsymbol{n}})\overline{E}^{-2}(\hat{\boldsymbol{n}})]\right). \qquad (20)
\end{aligned}$$

2.2 Normalization

$$[A^{EE}_\ell]^{-1} = \frac{1}{2}\int\frac{\mathrm{d}^2\boldsymbol{L}}{(2\pi)^2}\,\frac{1}{\hat{C}^{\rm EE}_L}\frac{1}{\hat{C}^{\rm EE}_{L'}}[(\boldsymbol{\ell}\odot_x\boldsymbol{L}\widetilde{C}^{\rm EE}_L+\boldsymbol{\ell}\odot_x\boldsymbol{L}'\widetilde{C}^{\rm EE}_{L'})\cos\varphi_{\boldsymbol{L}\boldsymbol{L}'}]^2 \qquad (21)$$

$$= \frac{1}{2}\int\frac{\mathrm{d}^2\boldsymbol{L}}{(2\pi)^2}\,\frac{1}{\hat{C}^{\rm EE}_L}\frac{1}{\hat{C}^{\rm EE}_{L'}}(\boldsymbol{\ell}\odot_x\boldsymbol{L}\widetilde{C}^{\rm EE}_L+\boldsymbol{\ell}\odot_x\boldsymbol{L}'\widetilde{C}^{\rm EE}_{L'})^2\left(\frac{\mathrm{e}^{2\mathrm{i}\varphi_{\boldsymbol{L}}}\,\mathrm{e}^{-2\mathrm{i}\varphi_{\boldsymbol{L}'}}+\mathrm{e}^{-2\mathrm{i}\varphi_{\boldsymbol{L}}}\,\mathrm{e}^{2\mathrm{i}\varphi_{\boldsymbol{L}'}}}{2}\right)^2 \qquad (22)$$

$$= \frac{1}{2}\int\frac{\mathrm{d}^2\boldsymbol{L}}{(2\pi)^2}\,\frac{1}{\hat{C}^{\rm EE}_L}\frac{1}{\hat{C}^{\rm EE}_{L'}}[(\boldsymbol{\ell}\odot_x\boldsymbol{L}\widetilde{C}^{\rm EE}_L)^2+(\boldsymbol{\ell}\odot_x\boldsymbol{L}'\widetilde{C}^{\rm EE}_{L'})^2+2\boldsymbol{\ell}\odot_x\boldsymbol{L}\widetilde{C}^{\rm EE}_L\boldsymbol{\ell}\odot_x\boldsymbol{L}'\widetilde{C}^{\rm EE}_{L'}]$$
$$\times\left(\frac{\mathrm{e}^{4\mathrm{i}\varphi_{\boldsymbol{L}}}\,\mathrm{e}^{-4\mathrm{i}\varphi_{\boldsymbol{L}'}}+\mathrm{e}^{-4\mathrm{i}\varphi_{\boldsymbol{L}}}\,\mathrm{e}^{4\mathrm{i}\varphi_{\boldsymbol{L}'}}+2}{4}\right) \qquad (23)$$

$$= \int\frac{\mathrm{d}^2\boldsymbol{L}}{(2\pi)^2}\,\frac{(\boldsymbol{\ell}\odot_x\boldsymbol{L}\widetilde{C}^{\rm EE}_L)^2+\boldsymbol{\ell}\odot_x\boldsymbol{L}\widetilde{C}^{\rm EE}_L\boldsymbol{\ell}\odot_x\boldsymbol{L}'\widetilde{C}^{\rm EE}_{L'}}{\hat{C}^{\rm EE}_L\hat{C}^{\rm EE}_{L'}}\left(\frac{\mathrm{e}^{4\mathrm{i}\varphi_{\boldsymbol{L}}}\,\mathrm{e}^{-4\mathrm{i}\varphi_{\boldsymbol{L}'}}+\mathrm{e}^{-4\mathrm{i}\varphi_{\boldsymbol{L}}}\,\mathrm{e}^{4\mathrm{i}\varphi_{\boldsymbol{L}'}}+2}{4}\right) \qquad (24)$$

$$= \int\frac{\mathrm{d}^2\boldsymbol{L}}{(2\pi)^2}\,\frac{(\boldsymbol{\ell}\odot_x\boldsymbol{L}\widetilde{C}^{\rm EE}_L)^2}{\hat{C}^{\rm EE}_L\hat{C}^{\rm EE}_{L'}}\left(\frac{\Re(\mathrm{e}^{4\mathrm{i}\varphi_{\boldsymbol{L}}}\,\mathrm{e}^{-4\mathrm{i}\varphi_{\boldsymbol{L}'}})+1}{2}\right)+\frac{\boldsymbol{\ell}\odot_x\boldsymbol{L}\widetilde{C}^{\rm EE}_L\boldsymbol{\ell}\odot_x\boldsymbol{L}'\widetilde{C}^{\rm EE}_{L'}}{\hat{C}^{\rm EE}_L\hat{C}^{\rm EE}_{L'}}\left(\frac{\mathrm{e}^{4\mathrm{i}\varphi_{\boldsymbol{L}}}\,\mathrm{e}^{-4\mathrm{i}\varphi_{\boldsymbol{L}'}}+1}{2}\right), \qquad (25)$$

3 Rotation and patchy tau

For patchy tau, the weight functions are given as

$$\begin{aligned}
f^{\tau,\Theta\Theta}_{\boldsymbol{\ell},\boldsymbol{L}} &= C^{\Theta\Theta}_L+C^{\Theta\Theta}_{L'}\,, \qquad (26)\\
f^{\tau,\Theta E}_{\boldsymbol{\ell},\boldsymbol{L}} &= C^{\Theta\rm E}_L\cos 2\varphi_{L,L'}+C^{\Theta\rm E}_{L'}\,, \qquad (27)\\
f^{\tau,\Theta B}_{\boldsymbol{\ell},\boldsymbol{L}} &= C^{\Theta\rm E}_L\sin 2\varphi_{L,L'}\,, \qquad (28)\\
f^{\tau,EE}_{\boldsymbol{\ell},\boldsymbol{L}} &= [C^{\rm EE}_L+C^{\rm EE}_{L'}]\cos 2\varphi_{L,L'}\,, \qquad (29)\\
f^{\tau,EB}_{\boldsymbol{\ell},\boldsymbol{L}} &= [C^{\rm EE}_L-C^{\rm BB}_{L'}]\sin 2\varphi_{L,L'}\,, \qquad (30)\\
f^{\tau,BB}_{\boldsymbol{\ell},\boldsymbol{L}} &= [C^{\rm BB}_L+C^{\rm BB}_{L'}]\cos 2\varphi_{L,L'}\,. \qquad (31)
\end{aligned}$$

Similarly, the weight function for polarization rotation is given by

$$f^{\alpha,\Theta\Theta}_{\boldsymbol{\ell},\boldsymbol{L}} = 0\,, \tag{32}$$
$$f^{\alpha,\Theta E}_{\boldsymbol{\ell},\boldsymbol{L}} = -2C^{\Theta\mathrm{E}}_L \sin 2\varphi_{L,L'}\,, \tag{33}$$
$$f^{\alpha,\Theta B}_{\boldsymbol{\ell},\boldsymbol{L}} = 2C^{\Theta\mathrm{E}}_L \cos 2\varphi_{L,L'}\,, \tag{34}$$
$$f^{\alpha,EE}_{\boldsymbol{\ell},\boldsymbol{L}} = -2[C^{\mathrm{EE}}_L + C^{\mathrm{EE}}_{L'}] \sin 2\varphi_{L,L'}\,, \tag{35}$$
$$f^{\alpha,EB}_{\boldsymbol{\ell},\boldsymbol{L}} = 2[C^{\mathrm{EE}}_L - C^{\mathrm{BB}}_{L'}] \cos 2\varphi_{L,L'}\,, \tag{36}$$
$$f^{\alpha,BB}_{\boldsymbol{\ell},\boldsymbol{L}} = -2[C^{\mathrm{BB}}_L + C^{\mathrm{BB}}_{L'}] \sin 2\varphi_{L,L'}\,. \tag{37}$$

We then find

$$f^{\alpha,XY}_{\boldsymbol{\ell},\boldsymbol{L}} = \frac{\partial}{\partial\varphi_{L,L'}} f^{\tau,XY}_{\boldsymbol{\ell},\boldsymbol{L}}\,. \tag{38}$$

3.1 Estimator

$$\bar{\tau}^{EB}_{\boldsymbol{\ell}} = \int\frac{\mathrm{d}^2\boldsymbol{L}}{(2\pi)^2}\,(\widetilde{C}^{\mathrm{EE}}_L - \widetilde{C}^{\mathrm{BB}}_{L'}) \sin 2\varphi_{\boldsymbol{L},\boldsymbol{L}'}\overline{E}_{\boldsymbol{L}}\overline{B}_{\boldsymbol{L}'} \tag{39}$$
$$= \int\frac{\mathrm{d}^2\boldsymbol{L}}{(2\pi)^2}\,(\widetilde{C}^{\mathrm{EE}}_L - \widetilde{C}^{\mathrm{BB}}_{L'})\Im[\mathrm{e}^{2\mathrm{i}\varphi_{\boldsymbol{L}}}\,\mathrm{e}^{-2\mathrm{i}\varphi_{\boldsymbol{L}'}}]\overline{E}_{\boldsymbol{L}}\overline{B}_{\boldsymbol{L}'} \tag{40}$$
$$= \int\frac{\mathrm{d}^2\boldsymbol{L}}{(2\pi)^2}\,\Im[(\widetilde{C}^{\mathrm{EE}}_L\,\mathrm{e}^{2\mathrm{i}\varphi_{\boldsymbol{L}}}\,\mathrm{e}^{-2\mathrm{i}\varphi_{\boldsymbol{L}'}} - \widetilde{C}^{\mathrm{BB}}_{L'}\,\mathrm{e}^{2\mathrm{i}\varphi_{\boldsymbol{L}}}\,\mathrm{e}^{-2\mathrm{i}\varphi_{\boldsymbol{L}'}})]\overline{E}_{\boldsymbol{L}}\overline{B}_{\boldsymbol{L}'}\,. \tag{41}$$

$$\bar{\alpha}^{EB}_{\boldsymbol{\ell}} = 2\int\frac{\mathrm{d}^2\boldsymbol{L}}{(2\pi)^2}\,\Re[(\widetilde{C}^{\mathrm{EE}}_L\,\mathrm{e}^{2\mathrm{i}\varphi_{\boldsymbol{L}}}\,\mathrm{e}^{-2\mathrm{i}\varphi_{\boldsymbol{L}'}} - \widetilde{C}^{\mathrm{BB}}_{L'}\,\mathrm{e}^{2\mathrm{i}\varphi_{\boldsymbol{L}}}\,\mathrm{e}^{-2\mathrm{i}\varphi_{\boldsymbol{L}'}})]\overline{E}_{\boldsymbol{L}}\overline{B}_{\boldsymbol{L}'}\,. \tag{42}$$

3.2 Normalization

$$[A^{\tau,EE}_\ell]^{-1} = \int\frac{\mathrm{d}^2\boldsymbol{L}}{(2\pi)^2}\,\frac{1}{2\hat{C}^{\mathrm{EE}}_L\hat{C}^{\mathrm{EE}}_{L'}}[(\widetilde{C}^{\mathrm{EE}}_L + \widetilde{C}^{\mathrm{EE}}_{L'})\cos 2\varphi_{\boldsymbol{L}\boldsymbol{L}'}]^2 \tag{43}$$
$$= \int\frac{\mathrm{d}^2\boldsymbol{L}}{(2\pi)^2}\,\frac{(\widetilde{C}^{\mathrm{EE}}_L + \widetilde{C}^{\mathrm{EE}}_{L'})^2}{2\hat{C}^{\mathrm{EE}}_L\hat{C}^{\mathrm{EE}}_{L'}}\left(\frac{\mathrm{e}^{2\mathrm{i}\varphi_{\boldsymbol{L}}}\,\mathrm{e}^{-2\mathrm{i}\varphi_{\boldsymbol{L}'}} + \mathrm{e}^{-2\mathrm{i}\varphi_{\boldsymbol{L}}}\,\mathrm{e}^{2\mathrm{i}\varphi_{\boldsymbol{L}'}}}{2}\right)^2 \tag{44}$$
$$= \frac{1}{4}\int\frac{\mathrm{d}^2\boldsymbol{L}}{(2\pi)^2}\,\frac{(\widetilde{C}^{\mathrm{EE}}_L + \widetilde{C}^{\mathrm{EE}}_{L'})^2}{\hat{C}^{\mathrm{EE}}_L\hat{C}^{\mathrm{EE}}_{L'}}\left(1 + \Re[\mathrm{e}^{4\mathrm{i}\varphi_{\boldsymbol{L}}}\,\mathrm{e}^{-4\mathrm{i}\varphi_{\boldsymbol{L}'}}]\right)\,, \tag{45}$$

$$[A^{\tau,EB}_\ell]^{-1} = \int\frac{\mathrm{d}^2\boldsymbol{L}}{(2\pi)^2}\,\frac{1}{\hat{C}^{\mathrm{EE}}_L\hat{C}^{\mathrm{BB}}_{L'}}[(\widetilde{C}^{\mathrm{EE}}_L - \widetilde{C}^{\mathrm{BB}}_{L'})\sin 2\varphi_{\boldsymbol{L}\boldsymbol{L}'}]^2 \tag{46}$$
$$= \int\frac{\mathrm{d}^2\boldsymbol{L}}{(2\pi)^2}\,\frac{(\widetilde{C}^{\mathrm{EE}}_L - \widetilde{C}^{\mathrm{BB}}_{L'})^2}{\hat{C}^{\mathrm{EE}}_L\hat{C}^{\mathrm{BB}}_{L'}}\left(\frac{\mathrm{e}^{2\mathrm{i}\varphi_{\boldsymbol{L}}}\,\mathrm{e}^{-2\mathrm{i}\varphi_{\boldsymbol{L}'}} - \mathrm{e}^{-2\mathrm{i}\varphi_{\boldsymbol{L}}}\,\mathrm{e}^{2\mathrm{i}\varphi_{\boldsymbol{L}'}}}{2\mathrm{i}}\right)^2 \tag{47}$$
$$= \frac{1}{2}\int\frac{\mathrm{d}^2\boldsymbol{L}}{(2\pi)^2}\,\frac{(\widetilde{C}^{\mathrm{EE}}_L - \widetilde{C}^{\mathrm{BB}}_{L'})^2}{\hat{C}^{\mathrm{EE}}_L\hat{C}^{\mathrm{BB}}_{L'}}\left(1 - \Re[\mathrm{e}^{4\mathrm{i}\varphi_{\boldsymbol{L}}}\,\mathrm{e}^{-4\mathrm{i}\varphi_{\boldsymbol{L}'}}]\right)\,, \tag{48}$$

$$[A^{\alpha,EB}_\ell]^{-1} = 2\int\frac{\mathrm{d}^2\boldsymbol{L}}{(2\pi)^2}\,\frac{(\widetilde{C}^{\mathrm{EE}}_L - \widetilde{C}^{\mathrm{BB}}_{L'})^2}{\hat{C}^{\mathrm{EE}}_L\hat{C}^{\mathrm{BB}}_{L'}}\left(1 + \Re[\mathrm{e}^{4\mathrm{i}\varphi_{\boldsymbol{L}}}\,\mathrm{e}^{-4\mathrm{i}\varphi_{\boldsymbol{L}'}}]\right)\,, \tag{49}$$

4 Disconnected four-point bias

4.1 RDN0

The RDN0 bias (after the mean-field bias correction) is defined as

$$\hat{N}_\ell^{XY,ZW} = \Gamma_\ell^{\rm XY,ZW} - N_\ell^{\rm XY,ZW} \,, \tag{50}$$

where

$$N_\ell^{\rm XY,ZW} = \frac{A_\ell^{\rm XY} A_\ell^{\rm ZW}}{\Delta^{\rm XY}\Delta^{\rm ZW}} \int\frac{{\rm d}^2\boldsymbol{L}}{(2\pi)^2} \int\frac{{\rm d}^2\boldsymbol{L}''}{(2\pi)^2}\, f_{\boldsymbol{\ell},\boldsymbol{L}}^{\rm XY} f_{\boldsymbol{\ell},\boldsymbol{L}'}^{\rm ZW} F_L^X F_{L'}^Z F_{L''}^Y F_{L'''}^W \\ \times [\langle X_{\boldsymbol{L}} Z^*_{\boldsymbol{L}''}\rangle\langle Y_{\boldsymbol{\ell}-\boldsymbol{L}} W^*_{\boldsymbol{\ell}-\boldsymbol{L}''}\rangle + \langle X_{\boldsymbol{L}} W^*_{\boldsymbol{\ell}-\boldsymbol{L}''}\rangle\langle Y_{\boldsymbol{\ell}-\boldsymbol{L}} Z^*_{\boldsymbol{L}''}\rangle] \,, \tag{51}$$

and

$$\Gamma_\ell^{\rm XY,ZW} = \frac{A_\ell^{\rm XY} A_\ell^{\rm ZW}}{\Delta^{\rm XY}\Delta^{\rm ZW}} \int\frac{{\rm d}^2\boldsymbol{L}}{(2\pi)^2} \int\frac{{\rm d}^2\boldsymbol{L}''}{(2\pi)^2}\, f_{\boldsymbol{\ell},\boldsymbol{L}}^{\rm XY} f_{\boldsymbol{\ell},\boldsymbol{L}''}^{\rm ZW} F_L^X F_{L'}^Z F_{L''}^Y F_{L''}^W \\ \times [\hat{X}_{\boldsymbol{L}} \widehat{Z}^*_{\boldsymbol{L}''}\langle Y_{\boldsymbol{\ell}-\boldsymbol{L}} W^*_{\boldsymbol{\ell}-\boldsymbol{L}''}\rangle + \langle X_{\boldsymbol{L}} Z^*_{\boldsymbol{L}''}\rangle \hat{Y}_{\boldsymbol{\ell}-\boldsymbol{L}} \widehat{W}^*_{\boldsymbol{\ell}-\boldsymbol{L}''} \\ + \hat{X}_{\boldsymbol{L}} \widehat{W}^*_{\boldsymbol{\ell}-\boldsymbol{L}''}\langle Y_{\boldsymbol{\ell}-\boldsymbol{L}} Z^*_{\boldsymbol{L}''}\rangle + \langle X_{\boldsymbol{L}} W^*_{\boldsymbol{\ell}-\boldsymbol{L}''}\rangle \hat{Y}_{\boldsymbol{\ell}-\boldsymbol{L}} \widehat{Z}^*_{\boldsymbol{L}''}] \,. \tag{52}$$

For example, if $X = Y = Z = W = \Theta$, we obtain

$$N_\ell^{\Theta\Theta,\Theta\Theta} = \frac{(A_\ell^{\Theta\Theta})^2}{2} \int\frac{{\rm d}^2\boldsymbol{L}}{(2\pi)^2} \int\frac{{\rm d}^2\boldsymbol{L}''}{(2\pi)^2}\, f_{\boldsymbol{\ell},\boldsymbol{L}}^{\Theta\Theta} f_{\boldsymbol{\ell},\boldsymbol{L}''}^{\Theta\Theta} F_L^\Theta F_{L'}^\Theta F_{L''}^\Theta F_{L'''}^\Theta \langle\Theta_{\boldsymbol{L}}\Theta^*_{\boldsymbol{L}''}\rangle\langle\Theta_{\boldsymbol{\ell}-\boldsymbol{L}}\Theta^*_{\boldsymbol{\ell}-\boldsymbol{L}''}\rangle \,, \tag{53}$$

$$\Gamma_\ell^{\Theta\Theta,\Theta\Theta} = 2\frac{(A_\ell^{\Theta\Theta})^2}{2} \int\frac{{\rm d}^2\boldsymbol{L}}{(2\pi)^2} \int\frac{{\rm d}^2\boldsymbol{L}''}{(2\pi)^2}\, f_{\boldsymbol{\ell},\boldsymbol{L}}^{\Theta\Theta} f_{\boldsymbol{\ell},\boldsymbol{L}''}^{\Theta\Theta} F_L^\Theta F_{L'}^\Theta F_{L''}^\Theta F_{L'''}^\Theta \widehat{\Theta}_{\boldsymbol{L}}\widehat{\Theta}^*_{\boldsymbol{L}''}\langle\Theta_{\boldsymbol{\ell}-\boldsymbol{L}}\Theta^*_{\boldsymbol{\ell}-\boldsymbol{L}''}\rangle \,. \tag{54}$$

4.2 Diagonal RDN0

Ignoring the off-diagonal elements of $\langle X_{\boldsymbol{L}} X'_{\boldsymbol{L}'}\rangle$, i.e., $X_{\boldsymbol{L}} Y^*_{\boldsymbol{L}'} \simeq \delta_{\boldsymbol{L},\boldsymbol{L}'} X_{\boldsymbol{L}} Y^*_{\boldsymbol{L}}$, we obtain the diagonal RDN0 as

$$\widehat{N}_\ell^{XY,ZW,{\rm diag}} = \Gamma_\ell^{\rm XY,ZW,diag} - N_\ell^{\rm XY,ZW,diag} \,, \tag{55}$$

where

$$N_\ell^{\rm XY,ZW,diag} = \frac{A_\ell^{\rm XY} A_\ell^{\rm ZW}}{\Delta^{\rm XY}\Delta^{\rm ZW}} \int\frac{{\rm d}^2\boldsymbol{L}}{(2\pi)^2} [f_{\boldsymbol{\ell},\boldsymbol{L}}^{\rm XY} f_{\boldsymbol{\ell},\boldsymbol{L}}^{\rm ZW} F_L^X F_L^Z F_{L'}^Y F_{L'}^W \hat{C}_L^{\rm XZ,th}\hat{C}_{L'}^{\rm YW,th} \\ + f_{\boldsymbol{\ell},\boldsymbol{L}}^{\rm XY} f_{\boldsymbol{\ell},\boldsymbol{L}'}^{\rm ZW} F_L^X F_L^W F_{L'}^Y F_{L'}^Z \hat{C}_L^{\rm XW,th}\hat{C}_{L'}^{\rm YZ,th}] \,. \tag{56}$$

and

$$\Gamma_\ell^{\rm XY,ZW,diag} = \frac{A_\ell^{\rm XY} A_\ell^{\rm ZW}}{\Delta^{\rm XY}\Delta^{\rm ZW}} \int\frac{{\rm d}^2\boldsymbol{L}}{(2\pi)^2} [f_{\boldsymbol{\ell},\boldsymbol{L}}^{\rm XY} f_{\boldsymbol{\ell},\boldsymbol{L}}^{\rm ZW} F_L^X F_L^Z F_{L'}^Y F_{L'}^W (\hat{C}_L^{\rm XZ}\hat{C}_{L'}^{\rm YW,th} + \hat{C}_L^{\rm XZ,th}\hat{C}_{L'}^{\rm YW}) \\ + f_{\boldsymbol{\ell},\boldsymbol{L}}^{\rm XY} f_{\boldsymbol{\ell},\boldsymbol{L}'}^{\rm ZW} F_L^X F_L^W F_{L'}^Y F_{L'}^Z (\hat{C}_L^{\rm XW}\hat{C}_{L'}^{\rm YZ,th} + \hat{C}_L^{\rm XW,th}\hat{C}_{L'}^{\rm YZ})] \,. \tag{57}$$

If $X = Y = Z = W = \Theta$, we obtain

$$N_\ell^{\Theta\Theta,\Theta\Theta,{\rm diag}} = \frac{(A_\ell^{\Theta\Theta})^2}{2} \int\frac{{\rm d}^2\boldsymbol{L}}{(2\pi)^2} (f_{\boldsymbol{\ell},\boldsymbol{L}}^{\Theta\Theta})^2 (F_L^\Theta F_{L'}^\Theta)^2 \widetilde{C}_L^{\Theta\Theta}\widetilde{C}_{L'}^{\Theta\Theta} \,, \tag{58}$$

$$\Gamma_\ell^{\Theta\Theta,\Theta\Theta,{\rm diag}} = 2\frac{(A_\ell^{\Theta\Theta})^2}{2} \int\frac{{\rm d}^2\boldsymbol{L}}{(2\pi)^2} (f_{\boldsymbol{\ell},\boldsymbol{L}}^{\Theta\Theta})^2 (F_L^\Theta F_{L'}^\Theta)^2 \hat{C}_L^{\Theta\Theta}\widetilde{C}_{L'}^{\Theta\Theta} \,. \tag{59}$$

To compute the above quantity, we define

$$\zeta_\ell^{XY,ZW}[\mathcal{C}] \equiv \int\frac{{\rm d}^2\boldsymbol{L}}{(2\pi)^2}\, f_{\boldsymbol{\ell},\boldsymbol{L}}^{\rm XY} f_{\boldsymbol{\ell},\boldsymbol{L}}^{\rm ZW} F_L^X F_L^Z F_{L'}^Y F_{L'}^W (\mathcal{C}_L^{XZ}\mathcal{C}_{L'}^{YW} + \mathcal{C}_L^{XW}\mathcal{C}_{L'}^{YW}) \,, \tag{60}$$

which is equivalent to the integral part of the normalization calculation if $XY = ZW$. Then, we obtain a computationally convenient form of the diagonal RDN0,

$$\widehat{N}_\ell^{XY,ZW,\mathrm{diag}} = \frac{A_\ell^{\mathrm{XY}} A_\ell^{\mathrm{ZW}}}{\Delta^{\mathrm{XY}}\Delta^{\mathrm{ZW}}} \left[-\zeta_\ell^{XY,ZW}[C-\widehat{C}] + \zeta_\ell^{XY,ZW}[\widehat{C}] \right] , \tag{61}$$

This form emerges naturally from the fact that the RDN0 does not have an error of $\delta C \equiv C - \widehat{C}$ at linear order.

References

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