Computing quadratic estimator, delensing in curvedsky

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Latest version

Abstract

Here, I describe an algorithm for computing the quadratic estimator and its normalization of the lensing, cosmic bi-refringence, patchy reionization, and so on.

Contents

1 Notations

In the followings, we use small letters for multipoles of the CMB anisotropies (e.g., ℓ), while large letters are used for multipoles of the distortion fields (lensing, rotation, etc).

1.1 CMB

Θ denotes the CMB temperature fluctuations, and Q and U denote the Stokes parameters of the CMB linear polarization. The following equation defines the harmonic coefficients of the temperature anisotropies (and, in general, any scalar quantities x):

$$x_{LM} = \int \mathrm{d}^2\hat{\boldsymbol{n}}\ Y^*_{LM}(\hat{\boldsymbol{n}})x(\hat{\boldsymbol{n}})\,. \tag{1}$$

where Y_{LM} is the spin-0 spherical harmonics. On the other hand, Q and U are changed by the rotation of the sphere, and are therefore usually transformed into the rotational invariant quantities, the E and B modes, as [1] [1]

$$[E \pm \mathrm{i}B]_{\ell m} = -\int \mathrm{d}^2\hat{\boldsymbol{n}}\ [Y^{\pm 2}_{\ell m}(\hat{\boldsymbol{n}})]^*[Q \pm \mathrm{i}U](\hat{\boldsymbol{n}})\,. \tag{2}$$

Here, $Y^{\pm 2}_{\ell m}$ is the spin-2 spherical harmonics. For short notation, we also use

$$\begin{aligned} \Xi^\pm &= E \pm \mathrm{i}B\,, \\ P^\pm &= Q \pm \mathrm{i}U \end{aligned} \tag{3}$$

1.2 Gravitational weak lensing

The lensing effect on CMB anisotropies is described as remapping of the unlensed CMB anisotropies by the deflection angle [2, 3]

$$X(\hat{\boldsymbol{n}}) = X[\hat{\boldsymbol{n}} + \boldsymbol{d}(\hat{\boldsymbol{n}})]\,, \tag{4}$$

[1] This definition is different from e.g. [2] by its sign.

where X is Θ or $P^{\pm}$. The deflection angle of the CMB lensing is decomposed into the lensing potential, ϕ, and curl mode, ϖ, as [4]

$$\boldsymbol{d}(\hat{\boldsymbol{n}}) = \boldsymbol{\nabla}\phi(\hat{\boldsymbol{n}}) + (\star\boldsymbol{\nabla})\varpi(\hat{\boldsymbol{n}})\,, \tag{5}$$

where the operator $\star\boldsymbol{\nabla}$ denotes the derivatives with 90° rotation counterclockwise on the plane perpendicular to the line-of-sight direction and then operation. The harmonic coefficients of ϕ and ϖ are given by Eq. (1). The remapping of the CMB anisotropies is then given by

$$X(\hat{\boldsymbol{n}}) = X(\hat{\boldsymbol{n}}) + [\boldsymbol{\nabla}\phi(\hat{\boldsymbol{n}}) + (\star\boldsymbol{\nabla})\varpi(\hat{\boldsymbol{n}})]\cdot\boldsymbol{\nabla}X + \mathcal{O}(\phi^2,\varpi^2)\,. \tag{6}$$

1.3 Polarization angle rotation

If the rotation angle is small, the modulation of polarization after a rotation by an angle α is given by (e.g. [5])

$$\delta P^{\pm}(\hat{\boldsymbol{n}}) = \pm 2\mathrm{i}\alpha(\hat{\boldsymbol{n}})P^{\pm}(\hat{\boldsymbol{n}})\,. \tag{7}$$

The harmonic coefficients of α is given by Eq. (1).

1.4 Amplitude modulations

Survey window, gain fluctuations, and the inhomogeneities of the reionization, could vary the amplitudes of the CMB fluctuations across the sky. Denoting the modulations as $1+\tau(\hat{\boldsymbol{n}})$, this leads to the modulation in CMB temperature and polarization as (e.g. [6])

$$\begin{aligned}\delta\Theta(\hat{\boldsymbol{n}}) &= \tau(\hat{\boldsymbol{n}})\Theta(\hat{\boldsymbol{n}})\,,\\ \delta P^{\pm}(\hat{\boldsymbol{n}}) &= \tau(\hat{\boldsymbol{n}})P^{\pm}(\hat{\boldsymbol{n}})\,.\end{aligned} \tag{8}$$

The harmonic coefficients of τ is given by Eq. (1).

1.5 Spherical Harmonics and Wigner-3j

The spherical harmonics is related to the Wigner-3j symbols as [7]

$$\int \mathrm{d}^2\hat{\boldsymbol{n}}\; Y^{s_1}_{\ell_1 m_1} Y^{s_2}_{\ell_2 m_2} Y^{s_3}_{\ell_3 m_3} = \sqrt{\frac{(2\ell_1+1)(2\ell_2+1)(2\ell_3+1)}{4\pi}} \begin{pmatrix}\ell_1 & \ell_2 & \ell_3\\ -s_1 & -s_2 & -s_3\end{pmatrix} \begin{pmatrix}\ell_1 & \ell_2 & \ell_3\\ m_1 & m_2 & m_3\end{pmatrix}\,, \tag{9}$$

with $s_1+s_2+s_3=0$ and $m_1+m_2+m_3=0$.

1.6 Derivatives of Spherical Harmonics

We first define the spin-operators acting on a spin-s quantity:

$$\begin{aligned}\eth &\equiv -\sin^s\theta\left(\frac{\partial}{\partial\theta}+\frac{\mathrm{i}}{\sin\theta}\frac{\partial}{\partial\varphi}\right)\sin^{-s}\theta = \frac{s}{\tan\theta}-\left(\frac{\partial}{\partial\theta}+\frac{\mathrm{i}}{\sin\theta}\frac{\partial}{\partial\varphi}\right)\,,\\ \bar{\eth} &\equiv -\sin^{-s}\theta\left(\frac{\partial}{\partial\theta}-\frac{\mathrm{i}}{\sin\theta}\frac{\partial}{\partial\varphi}\right)\sin^{s}\theta = \frac{-s}{\tan\theta}-\left(\frac{\partial}{\partial\theta}-\frac{\mathrm{i}}{\sin\theta}\frac{\partial}{\partial\varphi}\right)\,.\end{aligned} \tag{10}$$

The spin-weighted spherical harmonics are defined as:

$$\begin{aligned}Y^s_{\ell m} &= \left[\frac{(\ell-s)!}{(\ell+s)!}\right]^{1/2}\eth^s Y_{\ell m} \qquad (0\le s\le \ell)\\ &= \left[\frac{(\ell+s)!}{(\ell-s)!}\right]^{1/2}(-1)^s\bar{\eth}^s Y_{\ell m} \qquad (-\ell\le s\le 0)\,.\end{aligned} \tag{11}$$

The spherical harmonics, $Y_{\ell m}$, is a spin-0 function and $Y^s_{\ell m}$ has spin-s. Using the definition above, the derivative of the spin-weighted spherical harmonics is given by:

$$\eth Y^s_{\ell m} = \sqrt{(\ell-s)(\ell+s+1)}Y^{s+1}_{\ell m} = -\sqrt{2}a^s_\ell Y^{s+1}_{\ell m}\,, \tag{12}$$
$$\bar{\eth} Y^s_{\ell m} = -\sqrt{(\ell+s)(\ell-s+1)}Y^{s-1}_{\ell m} = \sqrt{2}a^{-s}_\ell Y^{s-1}_{\ell m}\,. \tag{13}$$

with $a^s_\ell = -\sqrt{(\ell-s)(\ell+s+1)/2}$. In the case if the spin operator acts on a spin-0 quantity, the spin operator is equivalent to:

$$\eth = \sqrt{2}\boldsymbol{e}\cdot\boldsymbol{\nabla}\,. \tag{14}$$

Here, we introduce the polarization vector $\boldsymbol{e}$ defined by

$$\boldsymbol{e} = \frac{\boldsymbol{e}_\theta + \mathrm{i}\boldsymbol{e}_\varphi}{\sqrt{2}}\,, \tag{15}$$

with $\boldsymbol{e}_i$ being the basis vectors orthogonal to the radial vector. The polarization vector satisfies $\boldsymbol{e}\cdot\boldsymbol{e}=0$, $\boldsymbol{e}\cdot\boldsymbol{e}^*=1$, $\star\boldsymbol{e}=-\mathrm{i}\boldsymbol{e}$. The covariant derivative to the spin-0 spherical harmonics is given by [8]:

$$\boldsymbol{\nabla}Y_{\ell m} = a^0_\ell\big(Y^1_{\ell m}\boldsymbol{e}^* - Y^{-1}_{\ell m}\boldsymbol{e}\big)\,. \tag{16}$$

Ignoring the numerical factor, the spin-weighted spherical harmonics is $Y^s_{\ell m} \sim \boldsymbol{e}^{a_1}\cdots\boldsymbol{e}^{a_s}\boldsymbol{\nabla}_{a_1}\cdots\boldsymbol{\nabla}_{a_s}Y_{\ell m}$.

For computing the lensing distortion on polarization below, it is convenient to regard the operation of $\boldsymbol{e}\cdot\boldsymbol{\nabla}$ as a spin operator [8]:

$$\boldsymbol{e}\cdot\boldsymbol{\nabla}Y^s_{\ell m} \equiv a^s_\ell Y^{s+1}_{\ell m}\,, \tag{17}$$
$$\boldsymbol{e}^*\cdot\boldsymbol{\nabla}Y^s_{\ell m} \equiv -a^{-s}_\ell Y^{s-1}_{\ell m}\,, \tag{18}$$

This is motivated by the fact that the lensing affects on the source function and the differential operation, $\boldsymbol{\nabla}$, appears before the multiplication of the polarization basis to the polarization tensor, $\boldsymbol{e}^a\boldsymbol{e}^b\boldsymbol{d}^c\boldsymbol{\nabla}_cP_{ab}$ which contains e.g. $\boldsymbol{e}^a\boldsymbol{e}^b\boldsymbol{e}^c\boldsymbol{\nabla}_c\boldsymbol{\nabla}_a\boldsymbol{\nabla}_bY_{\ell m}$ [2]. For $s=\pm2$, denoting $a^\pm = a^{\pm2}_\ell$, in the following calculation, we replace the covariant derivative to the spin-2 spherical harmonics with:

$$\boldsymbol{\nabla}Y^2_{\ell m} = a^+_\ell Y^3_{\ell m}\boldsymbol{e}^* - a^-_\ell Y^1_{\ell m}\boldsymbol{e}\,,$$
$$\boldsymbol{\nabla}Y^{-2}_{\ell m} = a^-_\ell Y^{-1}_{\ell m}\boldsymbol{e}^* - a^+_\ell Y^{-3}_{\ell m}\boldsymbol{e}\,. \tag{19}$$

1.7 Map derivatives

Derivative of scalar quantities such as the CMB temperature fluctuations and lensing potential is

$$\boldsymbol{\nabla}x = \sum_{LM}x_{LM}\boldsymbol{\nabla}Y_{LM} = \sum_{LM}x_{LM}a^0_L\left(Y^1_{LM}\boldsymbol{e}^* - Y^{-1}_{LM}\boldsymbol{e}\right) = x^+\boldsymbol{e}^* - x^-\boldsymbol{e}\,. \tag{20}$$

where we define

$$x^\pm \equiv \sum_{LM}x_{LM}a^0_LY^{\pm1}_{LM}\,, \tag{21}$$

and $(x^+)^* = -x^-$. The rotation of a pseudo-scalar quantity is given by

$$(\star\boldsymbol{\nabla})\varpi = \sum_{LM}\varpi_{LM}(\star\boldsymbol{\nabla})Y_{LM} = \sum_{LM}\varpi_{LM}a^0_L\mathrm{i}\left(Y^1_{LM}\boldsymbol{e}^* + Y^{-1}_{LM}\boldsymbol{e}\right) = \mathrm{i}(\varpi^+\boldsymbol{e}^* + \varpi^-\boldsymbol{e})\,, \tag{22}$$

and $(\varpi^+)^* = -\varpi^-$. Spin-2 fields such as the CMB linear polarization is given by

$$\boldsymbol{\nabla}P^+ = -\sum_{\ell m}\Xi^+_{\ell m}\boldsymbol{\nabla}Y^2_{\ell m} = -\sum_{\ell m}\Xi^+_{\ell m}\left(a^+_\ell Y^3_{\ell m}\boldsymbol{e}^* - a^-_\ell Y^1_{\ell m}\boldsymbol{e}\right) = -\Xi^{+^+}\boldsymbol{e}^* + \Xi^{+^-}\boldsymbol{e}\,, \tag{23}$$
$$\boldsymbol{\nabla}P^- = (\boldsymbol{\nabla}P^+)^* = -\sum_{\ell m}\Xi^-_{\ell m}\boldsymbol{\nabla}Y^{-2}_{\ell m} = -\sum_{\ell m}\Xi^-_{\ell m}\left(a^-_\ell Y^{-1}_{\ell m}\boldsymbol{e}^* - a^+_\ell Y^{-3}_{\ell m}\boldsymbol{e}\right) = -\Xi^{-^+}\boldsymbol{e}^* + \Xi^{-^-}\boldsymbol{e}\,. \tag{24}$$

Note that $(\Xi^{+^+})^* = -\Xi^{-^-}$ and $(\Xi^{+^-})^* = -\Xi^{-^+}$.

2 Distortion of CMB anisotropies

In the following, we first define useful quantities to compute the distortion effect. A multipole factor is defined as

$$\gamma_{\ell_1\ell_2\ell_3} \equiv \sqrt{\frac{(2\ell_1+1)(2\ell_2+1)(2\ell_3+1)}{4\pi}} \,. \tag{25}$$

The convolution operator in full sky is defined as

$$\widetilde{\sum_{LM\ell'm'}}^{(\ell m)} \equiv \sum_{LM\ell'm'} (-1)^m \begin{pmatrix} \ell & L & \ell' \\ -m & M & m' \end{pmatrix} \,. \tag{26}$$

We introduce the following coefficients;

$$c_\phi = 1\,, \tag{27}$$
$$c_\varpi = -\mathrm{i}\,, \tag{28}$$
$$c_\alpha = 1\,, \tag{29}$$
$$c_\tau = 1\,, \tag{30}$$

and

$$\zeta^+ = 1\,, \tag{31}$$
$$\zeta^- = \mathrm{i}\,. \tag{32}$$

Parity symmetry indicators are given by

$$p_{\ell_1\ell_2\ell_3} \equiv (-1)^{\ell_1+\ell_2+\ell_3}\,, \tag{33}$$
$$q^{x,\pm}_{\ell_1\ell_2\ell_3} \equiv c_x\frac{1\pm c_x^2(-1)^{\ell_1+\ell_2+\ell_3}}{2}\,, \tag{34}$$
$$q^{\pm}_{\ell_1\ell_2\ell_3} \equiv \frac{1\pm(-1)^{\ell_1+\ell_2+\ell_3}}{2}\,. \tag{35}$$

2.1 Lensing distortion

The lensing contributions in the position space become

$$\begin{aligned}
\delta^\phi\Theta &= \boldsymbol{\nabla}\phi\cdot\boldsymbol{\nabla}\Theta = -\phi^-\Theta^+ - \phi^+\Theta^-\,, \\
\delta^\varpi\Theta &= (\star\boldsymbol{\nabla})\varpi\cdot\boldsymbol{\nabla}\Theta = \mathrm{i}(\varpi^-\Theta^+ - \varpi^+\Theta^-)\,, \\
\delta^\phi P^\pm &= \boldsymbol{\nabla}\phi\cdot\boldsymbol{\nabla}P^\pm = \phi^-\Xi^{\pm^+} + \phi^+\Xi^{\pm^-}\,, \\
\delta^\varpi P^\pm &= (\star\boldsymbol{\nabla})\varpi\cdot\boldsymbol{\nabla}P^\pm = \mathrm{i}(-\varpi^-\Xi^{\pm^+} + \varpi^+\Xi^{\pm^-})\,.
\end{aligned} \tag{36}$$

The spherical harmonic transform of the lensing contributions in temperature is

$$\begin{aligned}
\delta\Theta_{\ell m} &= -c_x\int \mathrm{d}^2\hat{\boldsymbol{n}}\; Y^*_{\ell m}[x^-\Theta^+ + c_x^2 x^+\Theta^-] \\
&= -\sum_{LM\ell'm'} x_{LM}\Theta_{\ell'm'}a^0_L a^0_{\ell'} c_x \int \mathrm{d}^2\hat{\boldsymbol{n}}\; (-1)^m Y_{\ell,-m}[Y^{-1}_{LM}Y^1_{\ell'm'} + c_x^2 Y^1_{LM}Y^{-1}_{\ell'm'}] \\
&= -\sum_{LM\ell'm'} x_{LM}\Theta_{\ell'm'}a^0_L a^0_{\ell'} 2q^{x,+}_{\ell L\ell'}\gamma_{\ell L\ell'}(-1)^m \begin{pmatrix} \ell & L & \ell' \\ -m & M & m' \end{pmatrix}\begin{pmatrix} \ell & L & \ell' \\ 0 & 1 & -1 \end{pmatrix} \\
&= -\widetilde{\sum_{LM\ell'm'}}^{(\ell m)} x_{LM}\Theta_{\ell'm'}a^0_L a^0_{\ell'} 2q^{x,+}_{\ell L\ell'}\gamma_{\ell L\ell'} \begin{pmatrix} \ell & L & \ell' \\ 0 & 1 & -1 \end{pmatrix} \\
&= \widetilde{\sum_{LM\ell'm'}}^{(\ell m)} x_{LM}\Theta_{\ell'm'}W^{x,0}_{\ell L\ell'}\,.
\end{aligned} \tag{37}$$

Here, we denote

$$W^{x,0}_{\ell_1\ell_2\ell_3} = -2a^0_{\ell_2}a^0_{\ell_3}q^{x,+}_{\ell_1\ell_2\ell_3}\gamma_{\ell_1\ell_2\ell_3}\begin{pmatrix}\ell_1 & \ell_2 & \ell_3 \\ 0 & 1 & -1\end{pmatrix} . \quad (38)$$

Here, $(W^{\phi,0}_{\ell_1\ell_2\ell_3})^* = W^{\phi,0}_{\ell_1\ell_2\ell_3}$. The above quantity is consistent with Ref. [9] and also $(W^{\varpi,0}_{\ell_1\ell_2\ell_3})^* = p_{\ell_1\ell_2\ell_3}W^{\varpi,0}_{\ell_1\ell_2\ell_3}$. Note that

$$(-1)^m\begin{pmatrix}\ell & L & \ell' \\ -m & M & m'\end{pmatrix}W^{\phi,0}_{\ell L\ell'} = \int \mathrm{d}^2\hat{\boldsymbol{n}}\ Y^*_{\ell m}(\boldsymbol{\nabla}Y_{LM})\cdot\boldsymbol{\nabla}Y_{\ell'm'} , \quad (39)$$

$$(-1)^m\begin{pmatrix}\ell & L & \ell' \\ -m & M & m'\end{pmatrix}W^{\varpi,0}_{\ell L\ell'} = \int \mathrm{d}^2\hat{\boldsymbol{n}}\ Y^*_{\ell m}[(\star\boldsymbol{\nabla})Y_{LM}]\cdot\boldsymbol{\nabla}Y_{\ell'm'} . \quad (40)$$

On the other hand, the lensed polarization anisotropies are given by

$$\begin{aligned}\delta\Xi^+_{\ell m} &= c_x\int \mathrm{d}^2\hat{\boldsymbol{n}}\ (Y^2_{\ell m})^*[x^-\Xi^{+^+} + c_x^2 x^+\Xi^{+^-}] \\ &= -c_x\sum_{LM\ell'm'}x_{LM}\Xi^+_{\ell'm'}a^0_L\int \mathrm{d}^2\hat{\boldsymbol{n}}\ (Y^2_{\ell m})^*[Y^{-1}_{LM}a^+_{\ell'}Y^3_{\ell'm'} + c_x^2Y^1_{LM}a^-_{\ell'}Y^1_{\ell'm'}] \\ &= -c_x\sum_{LM\ell'm'}(-1)^m\begin{pmatrix}\ell & L & \ell' \\ -m & M & m'\end{pmatrix} \\ &\quad\times x_{LM}\Xi^+_{\ell'm'}\gamma_{\ell L\ell'}a^0_L\left[a^+_{\ell'}\begin{pmatrix}\ell & L & \ell' \\ 2 & 1 & -3\end{pmatrix} + c_x^2a^-_{\ell'}\begin{pmatrix}\ell & L & \ell' \\ 2 & -1 & -1\end{pmatrix}\right] \\ &= \widetilde{\sum_{LM\ell'm'}}^{(\ell m)}x_{LM}\Xi^+_{\ell'm'}W^{x,+2}_{\ell L\ell'} ,\end{aligned} \quad (41)$$

with

$$W^{x,2}_{\ell_1\ell_2\ell_3} = -c_x\gamma_{\ell_1\ell_2\ell_3}a^0_{\ell_2}\left[a^+_{\ell_3}\begin{pmatrix}\ell_1 & \ell_2 & \ell_3 \\ 2 & 1 & -3\end{pmatrix} + c_x^2a^-_{\ell_3}\begin{pmatrix}\ell_1 & \ell_2 & \ell_3 \\ 2 & -1 & -1\end{pmatrix}\right] . \quad (42)$$

Similarly, we have

$$\begin{aligned}\delta\Xi^-_{\ell m} &= c_x\int \mathrm{d}^2\hat{\boldsymbol{n}}\ (Y^2_{\ell m})^*[x^-\Xi^{-^+} + c_x^2 x^+\Xi^{-^-}] \\ &= -c_x\sum_{LM\ell'm'}x_{LM}\Xi^-_{\ell'm'}a^0_L\int \mathrm{d}^2\hat{\boldsymbol{n}}\ (Y^{-2}_{\ell m})^*[Y^{-1}_{LM}a^-_{\ell'}Y^{-1}_{\ell'm'} + c_x^2Y^1_{LM}a^+_{\ell'}Y^{-3}_{\ell'm'}] \\ &= -c_x\sum_{LM\ell'm'}(-1)^m\begin{pmatrix}\ell & L & \ell' \\ -m & M & m'\end{pmatrix} \\ &\quad\times x_{LM}\Xi^-_{\ell'm'}\gamma_{\ell L\ell'}a^0_L\left[a^-_{\ell'}\begin{pmatrix}\ell & L & \ell' \\ -2 & 1 & 1\end{pmatrix} + c_x^2a^+_{\ell'}\begin{pmatrix}\ell & L & \ell' \\ -2 & -1 & 3\end{pmatrix}\right] \\ &= \widetilde{\sum_{LM\ell'm'}}^{(\ell m)}x_{LM}\Xi^-_{\ell'm'}W^{x,-2}_{\ell L\ell'} ,\end{aligned} \quad (43)$$

with

$$W^{x,-2}_{\ell_1\ell_2\ell_3} = p_{\ell_1\ell_2\ell_3}c_x^2W^{x,2}_{\ell_1\ell_2\ell_3} . \quad (44)$$

Note that

$$(-1)^m\begin{pmatrix}\ell & L & \ell' \\ -m & M & m'\end{pmatrix}W^{\phi,\pm2}_{\ell L\ell'} = \int \mathrm{d}^2\hat{\boldsymbol{n}}\ (Y^{\pm2}_{\ell m})^*(\boldsymbol{\nabla}Y_{LM})\cdot\boldsymbol{\nabla}Y^{\pm2}_{\ell'm'} , \quad (45)$$

$$(-1)^m\begin{pmatrix}\ell & L & \ell' \\ -m & M & m'\end{pmatrix}W^{\varpi,\pm2}_{\ell L\ell'} = \int \mathrm{d}^2\hat{\boldsymbol{n}}\ (Y^{\pm2}_{\ell m})^*[(\star\boldsymbol{\nabla})Y_{LM}]\cdot\boldsymbol{\nabla}Y^{\pm2}_{\ell'm'} . \quad (46)$$

2.2 Rotation distortion

The E and B modes after the rotation are given by

$$\begin{aligned}\delta\Xi^{\pm}_{\ell m} &= \mp 2\mathrm{i}\int \mathrm{d}^2\hat{\boldsymbol{n}}\ (Y^{\pm 2}_{\ell m})^*\alpha P^{\pm} \\ &= \pm 2\mathrm{i}\sum_{LM\ell'm'}\alpha_{LM}\Xi^{\pm}_{\ell'm'}\int \mathrm{d}^2\hat{\boldsymbol{n}}\ (Y^{\pm 2}_{\ell m})^* Y_{LM}Y^{\pm 2}_{\ell'm'} \\ &= \pm 2\mathrm{i}\sum_{LM\ell'm'}(-1)^m\begin{pmatrix}\ell & L & \ell' \\ -m & M & m'\end{pmatrix}\alpha_{LM}\Xi^{\pm}_{\ell'm'}\gamma_{\ell L\ell'}\begin{pmatrix}\ell & L & \ell' \\ \pm 2 & 0 & \mp 2\end{pmatrix} \\ &= \widetilde{\sum_{LM\ell'm'}}^{(\ell m)}\alpha_{LM}\Xi^{\pm}_{\ell'm'}W^{\alpha,\pm 2}_{\ell L\ell'}\,, \end{aligned} \tag{47}$$

with

$$W^{\alpha,\pm 2}_{\ell_1\ell_2\ell_3} = \pm 2\mathrm{i}\gamma_{\ell_1\ell_2\ell_3}\begin{pmatrix}\ell_1 & \ell_2 & \ell_3 \\ \pm 2 & 0 & \mp 2\end{pmatrix}\,. \tag{48}$$

2.3 Amplitude distortion

The harmonics transform of $\epsilon(\hat{\boldsymbol{n}})\Theta(\hat{\boldsymbol{n}})$ is

$$\begin{aligned}\delta\Theta_{\ell m} &= \int \mathrm{d}^2\hat{\boldsymbol{n}}\ Y^*_{\ell m}\epsilon(\hat{\boldsymbol{n}})\Theta(\hat{\boldsymbol{n}}) \\ &= \sum_{LM\ell'm'}\epsilon_{LM}\Theta_{\ell'm'}\int \mathrm{d}^2\hat{\boldsymbol{n}}\ Y^*_{\ell m}Y_{LM}Y_{\ell'm'} \end{aligned} \tag{49}$$

$$\begin{aligned} &= \sum_{LM\ell'm'}\epsilon_{LM}\Theta_{\ell'm'}q^{+}_{\ell L\ell'}\gamma_{\ell L\ell'}(-1)^m\begin{pmatrix}\ell & L & \ell' \\ -m & M & m'\end{pmatrix}\begin{pmatrix}\ell & L & \ell' \\ 0 & 0 & 0\end{pmatrix} \\ &= \widetilde{\sum_{LM\ell'm'}}^{(\ell m)}\epsilon_{LM}\Theta_{\ell'm'}W^{\epsilon,0}_{\ell L\ell'}\,, \end{aligned} \tag{50}$$

where

$$W^{\epsilon,0}_{\ell L\ell'} = q^{+}_{\ell L\ell'}\gamma_{\ell L\ell'}\begin{pmatrix}\ell & L & \ell' \\ 0 & 0 & 0\end{pmatrix} = \gamma_{\ell L\ell'}\begin{pmatrix}\ell & L & \ell' \\ 0 & 0 & 0\end{pmatrix}\,. \tag{51}$$

The polarization anisotropies with the amplitude distortion are given by

$$\delta\Xi^{\pm}_{\ell m} = -\int \mathrm{d}^2\hat{\boldsymbol{n}}\ (Y^{\pm 2}_{\ell m})^*\epsilon(\hat{\boldsymbol{n}})P^{\pm}(\hat{\boldsymbol{n}}) = \widetilde{\sum_{LM\ell'm'}}^{(\ell m)}\epsilon_{LM}\Xi^{\pm}_{\ell'm'}W^{\epsilon,\pm 2}_{\ell L\ell'}\,, \tag{52}$$

with

$$W^{\epsilon,\pm 2}_{\ell_1\ell_2\ell_3} = \gamma_{\ell_1\ell_2\ell_3}\begin{pmatrix}\ell_1 & \ell_2 & \ell_3 \\ \pm 2 & 0 & \mp 2\end{pmatrix}\,. \tag{53}$$

2.4 General form

Consider the following more general linear distortion:

$$\delta\Theta_{\ell m} = \sum_{LM}x^0_{LM}\frac{\partial\Theta_{\ell m}}{\partial x^0_{LM}} \tag{54}$$

$$\delta\Xi^{\pm}_{\ell m} = \sum_{LM}x^{\pm}_{LM}\frac{\partial\Xi^{\pm}_{\ell m}}{\partial x^{\pm}_{LM}}\,, \tag{55}$$

For example, Eq. (49) leads to:

$$\frac{\partial\Theta_{\ell m}}{\partial\epsilon_{LM}} = \sum_{\ell'm'}\Theta_{\ell'm'}\int \mathrm{d}^2\hat{\boldsymbol{n}}\; Y^*_{\ell m}Y_{LM}Y_{\ell'm'} \tag{56}$$

$$\frac{\partial\Xi^\pm_{\ell m}}{\partial\epsilon_{LM}} = \sum_{\ell'm'}\Xi^\pm_{\ell'm'}\int \mathrm{d}^2\hat{\boldsymbol{n}}\; (Y^{\pm2}_{\ell m})^*Y_{LM}Y^{\pm2}_{\ell'm'}\,, \tag{57}$$

2.5 Translate into E/B

Now we consider the distorted E/B modes separately. In general, if the distortion is given by

$$\delta\Xi^\pm_{\ell m} = \widetilde{\sum_{LM\ell'm'}}^{(\ell m)} x_{LM}\Xi^\pm_{\ell'm'}W^{x,\pm2}_{\ell L\ell'}\,, \tag{58}$$

we obtain

$$\delta E_{\ell m} = \widetilde{\sum_{LM\ell'm'}}^{(\ell m)} x_{LM}\left(E_{\ell'm'}W^{x,+}_{\ell L\ell'} + B_{\ell'm'}W^{x,-}_{\ell L\ell'}\right)\,, \tag{59}$$

$$\delta B_{\ell m} = \widetilde{\sum_{LM\ell'm'}}^{(\ell m)} x_{LM}\left(-E_{\ell'm'}W^{x,-}_{\ell L\ell'} + B_{\ell'm'}W^{x,+}_{\ell L\ell'}\right)\,, \tag{60}$$

where we define

$$W^{x,\pm}_{\ell L\ell'} \equiv \zeta^\pm\frac{W^{x,+2}_{\ell L\ell'} \pm W^{x,-2}_{\ell L\ell'}}{2}\,. \tag{61}$$

For lensing, the functional form of W is given by

$$\begin{aligned} W^{x,\pm}_{\ell_1\ell_2\ell_3} &= \zeta^\pm\frac{1\pm c_x^2(-1)^{\ell_1+\ell_2+\ell_3}}{2}W^{x,2}_{\ell_1\ell_2\ell_3} \\ &= -\zeta^\pm q^{x,\pm}_{\ell_1\ell_2\ell_3}\gamma_{\ell_1\ell_2\ell_3}a^0_{\ell_2}\left[a^+_{\ell_3}\begin{pmatrix}\ell_1 & \ell_2 & \ell_3\\ 2 & 1 & -3\end{pmatrix} + c_x^2 a^-_{\ell_3}\begin{pmatrix}\ell_1 & \ell_2 & \ell_3\\ 2 & -1 & -1\end{pmatrix}\right]\,. \end{aligned} \tag{62}$$

For polarization rotation, we obtain

$$W^{\alpha,\pm}_{\ell_1\ell_2\ell_3} = 2\mathrm{i}\zeta^\pm q^{\mp}_{\ell_1\ell_2\ell_3}\gamma_{\ell_1\ell_2\ell_3}\begin{pmatrix}\ell_1 & \ell_2 & \ell_3\\ 2 & 0 & -2\end{pmatrix}\,. \tag{63}$$

This is consistent with [10] in the absence of B-modes. For amplitude modulations, we find

$$W^{\epsilon,\pm}_{\ell_1\ell_2\ell_3} = \zeta^\pm q^{\pm}_{\ell_1\ell_2\ell_3}\gamma_{\ell_1\ell_2\ell_3}\begin{pmatrix}\ell_1 & \ell_2 & \ell_3\\ 2 & 0 & -2\end{pmatrix}\,. \tag{64}$$

2.6 Properties of weight functions

2.6.1 Parity symmetry

The property of W is also important. If x is parity even, $W^{x,+}_{\ell L\ell'}$ and $W^{x,-}_{\ell L\ell'}$ are non-zero only when $\ell+L+\ell'$ is even and odd, respectively. If x is parity odd, $W^{x,-}_{\ell L\ell'}$ and $W^{x,+}_{\ell L\ell'}$ are non-zero only when $\ell+L+\ell'$ is even and odd, respectively. $W^{x,0}$ is the same as $W^{x,+}$.

2.6.2 The lowest multipole of distortion fields

The above parity symmetry limits the possible lowest multipole of the distortion fields contained in the CMB anisotropies. For example, the dipole of the lensing potential, ϕ_{1M}, does not mix different modes in ΘB and EB correlations because of the property of the Wigner-3j symbols and the rotational invariance of CMB fields as follows. First, without loss of generality, we can choose the z-axis for defining the spherical harmonics so that $M = 0$ for the dipole fluctuations, ϕ_{1M}, by rotating the unit sphere, leading to $m = m'$ from $M + m - m' = 0$. The weight function, $W^{\phi,-}_{\ell 1 \ell'}$ contains the Wigner-3j symbols given in Eq. (62). To satisfy the triangular condition, $|\ell - \ell'| \leq 1 \leq \ell + \ell'$, for $\ell + 1 + \ell'$ is odd, we need $\ell = \ell'$. Thus, for $L = 1$, ΘB and EB does not have mode mixing. This also holds for even parity distortion fields, such as the amplitude modulation. This means that the ΘB and EB quadratic estimators cannot reconstruct $L = 1$ of even parity distortion fields. On the other hand, for the parity odd distortion fields, the situation is opposite and only ΘB and EB quadratic estimators can reconstruct $L = 1$ modes.

2.6.3 Dual relationships

The weight of the lensing and imaginary lensing has a relationship due to its real-imaginary conjugate;

$$W^{\tilde{x},+}_{\ell_1\ell_2\ell_3} = 2W^{x,-}_{\ell_1\ell_2\ell_3} \,, \tag{65}$$

$$W^{\tilde{x},-}_{\ell_1\ell_2\ell_3} = -2W^{x,+}_{\ell_1\ell_2\ell_3} \,. \tag{66}$$

Similarly, we have

$$W^{\alpha,+}_{\ell_1\ell_2\ell_3} = 2W^{\tau,-}_{\ell_1\ell_2\ell_3} \,, \tag{67}$$

$$W^{\alpha,-}_{\ell_1\ell_2\ell_3} = -2W^{\tau,+}_{\ell_1\ell_2\ell_3} \,, \tag{68}$$

and

$$W^{\alpha,\pm}_{\ell_3\ell_2\ell_1} = W^{\alpha,\pm}_{\ell_1\ell_2\ell_3} \,, \tag{69}$$

$$W^{\epsilon,s}_{\ell_3\ell_2\ell_1} = W^{\epsilon,s}_{\ell_1\ell_2\ell_3} \,, \tag{70}$$

where $s = 0, \pm$.

2.7 Summary

The above all distortions are described in the following form:

$$\delta\Theta_{\ell m} = \widetilde{\sum_{LM\ell' m'}}^{(\ell m)} x_{LM}\Theta_{\ell' m'} W^{x,0}_{\ell L \ell'} \,, \tag{71}$$

$$\delta E_{\ell m} = \widetilde{\sum_{LM\ell' m'}}^{(\ell m)} x_{LM} \left(E_{\ell' m'} W^{x,+}_{\ell L \ell'} + B_{\ell' m'} W^{x,-}_{\ell L \ell'} \right) \,, \tag{72}$$

$$\delta B_{\ell m} = \widetilde{\sum_{LM\ell' m'}}^{(\ell m)} x_{LM} \left(-E_{\ell' m'} W^{x,-}_{\ell L \ell'} + B_{\ell' m'} W^{x,+}_{\ell L \ell'} \right) \tag{73}$$

where x is a distortion field.

3 Quadratic estimator

3.1 Distortion induced anisotropies

The distortion fields x described above induce the off-diagonal elements of the covariance ($\ell \neq \ell'$ or $m \neq m'$), [11, 12]

$$\langle \widetilde{X}_{\ell m} \widetilde{Y}_{\ell' m'} \rangle_{\rm CMB} = \sum_{LM} \begin{pmatrix} \ell & \ell' & L \\ m & m' & M \end{pmatrix} f^{x,({\rm XY})}_{\ell L \ell'} x^*_{LM} \,, \tag{74}$$

where $\langle \cdots \rangle_{\rm CMB}$ denotes the ensemble average over the primary CMB anisotropies with a fixed realization of the distortion fields. We ignore the higher-order terms of the distortion fields. The functional form of the weight functions f are summarized in Sec. 3.3. Note that

$$\langle \widetilde{Y}_{\ell m} \widetilde{X}_{\ell' m'} \rangle_{\rm CMB} = \sum_{LM} \begin{pmatrix} \ell' & \ell & L \\ m' & m & M \end{pmatrix} f^{x,({\rm XY})}_{\ell' L \ell} x^*_{LM} = \sum_{LM} \begin{pmatrix} \ell & \ell' & L \\ m & m' & M \end{pmatrix} p_{\ell\ell' L} f^{x,({\rm XY})}_{\ell' L \ell} x^*_{LM} \,, \tag{75}$$

and we obtain

$$f^{x,({\rm YX})}_{\ell L \ell'} = p_{\ell \ell' L} f^{x,({\rm XY})}_{\ell' L \ell} \,. \tag{76}$$

A more general form is given by using Eqs. (54) and (55):

$$\langle \widetilde{X}_{\ell m} \widetilde{Y}_{\ell' m'} \rangle_{\rm CMB} = \sum_{LM} x^*_{LM} \left\langle \widehat{X}_{\ell m} \frac{\partial \widehat{Y}_{\ell' m'}}{\partial x^*_{LM}} + \widehat{Y}_{\ell' m'} \frac{\partial \widehat{X}_{\ell m}}{\partial x^*_{LM}} \right\rangle_{\rm CMB} \,, \tag{77}$$

Note that the bispectrum between the observed CMB anisotropies and distortion fields become:

$$\langle x_{LM} \langle \widetilde{X}_{\ell m} \widetilde{Y}_{\ell' m'} \rangle_{\rm CMB} \rangle = C^{xx}_L \left\langle \widehat{X}_{\ell m} \frac{\partial \widehat{Y}_{\ell' m'}}{\partial x^*_{LM}} + \widehat{Y}_{\ell' m'} \frac{\partial \widehat{X}_{\ell m}}{\partial x^*_{LM}} \right\rangle_{\rm CMB} \,. \tag{78}$$

The rotational invariance of the bispectrum limits the possible combination of the three multipoles which leads to the Wigner 3j symbols:

$$\left\langle \widehat{X}_{\ell m} \frac{\partial \widehat{Y}_{\ell' m'}}{\partial x^*_{LM}} + \widehat{Y}_{\ell' m'} \frac{\partial \widehat{X}_{\ell m}}{\partial x^*_{LM}} \right\rangle_{\rm CMB} = \frac{\partial \langle \widehat{X}_{\ell m} \widehat{Y}_{\ell' m'} \rangle_{\rm CMB}}{\partial x^*_{LM}} \equiv \begin{pmatrix} \ell' & \ell & L \\ m' & m & M \end{pmatrix} f^{x,({\rm XY})}_{\ell' L \ell} \tag{79}$$

The violation of the statistical isotropy, therefore, leads to more complicated form of the covariance.

3.2 Quadratic estimator

With a quadratic combination of observed CMB anisotropies, $\widehat{X}$ and $\widehat{Y}$, the general quadratic estimators are formed as

$$[\widehat{x}^{\rm XY}_{LM}]^* = A^{x,({\rm XY})}_L \sum_{\ell\ell' mm'} \begin{pmatrix} \ell & \ell' & L \\ m & m' & M \end{pmatrix} g^{x,({\rm XY})}_{\ell L \ell'} \widehat{X}_{\ell m} \widehat{Y}_{\ell' m'} \,. \tag{80}$$

Here we define

$$g^{x,({\rm XY})}_{\ell L \ell'} = \frac{[f^{x,({\rm XY})}_{\ell L \ell'}]^*}{\Delta^{\rm XY} \widehat{C}^{\rm XX}_\ell \widehat{C}^{\rm YY}_{\ell'}} \,, \tag{81}$$

$$[A^{x,({\rm XY})}_L]^{-1} = \frac{1}{2L+1} \sum_{\ell\ell'} f^{x,({\rm XY})}_{\ell L \ell'} g^{x,({\rm XY})}_{\ell L \ell'} \,, \tag{82}$$

where $\Delta^{\rm XX} = 2$, $\Delta^{\rm EB} = \Delta^{\rm TB} = 1$, and $\widehat{C}^{\rm XX}_\ell$ ($\widehat{C}^{\rm YY}_\ell$) is the observed power spectrum.

3.3 Weight Function: Derivations

3.3.1 $\Theta\Theta$

Let us first consider the temperature case. There are two contributions to the temperature quadratic estimator, and the one is given as

$$\begin{aligned}\langle(\delta\Theta_{\ell m})\Theta_{\ell'm'}\rangle &= \sum_{LM\ell''m''}(-1)^m\begin{pmatrix}\ell & L & \ell''\\ -m & M & m''\end{pmatrix}x_{LM}W^{x,0}_{\ell L\ell''}\langle\Theta_{\ell''m''}\Theta_{\ell'm'}\rangle \\ &= \sum_{LM\ell''m''}(-1)^m\begin{pmatrix}\ell & L & \ell''\\ -m & M & m''\end{pmatrix}x_{LM}W^{x,0}_{\ell L\ell'}\delta_{\ell''\ell'}\delta_{m'',-m'}(-1)^{m'}C^{\Theta\Theta}_{\ell'} \\ &= \sum_{LM}(-1)^{m+m'}\begin{pmatrix}\ell & L & \ell'\\ -m & M & -m'\end{pmatrix}x_{LM}W^{x,0}_{\ell L\ell'}C^{\Theta\Theta}_{\ell'} \\ &= \sum_{LM}\begin{pmatrix}\ell & \ell' & L\\ m & m' & M\end{pmatrix}x^*_{LM}W^{x,0}_{\ell L\ell'}C^{\Theta\Theta}_{\ell'}\,. \end{aligned} \tag{83}$$

In the above, from the third to the last equation, we use $m+m'=-M$, change the sign of $m,m'M$, $M\to -M$, and further change the order of column in the Wigner 3j. The other term is obtained by $(\ell'',m'')\leftrightarrow(\ell,m)$ and is given by

$$\langle\Theta_{\ell m}\delta\Theta_{\ell'm'}\rangle = \sum_{LM}p_{\ell\ell'L}\begin{pmatrix}\ell & \ell' & L\\ m & m' & M\end{pmatrix}x^*_{LM}W^{x,0}_{\ell'L\ell}C^{\Theta\Theta}_{\ell}\,. \tag{84}$$

The sum of the above two equations yield

$$f^{x,(\Theta\Theta)}_{\ell L\ell'} = W^{x,0}_{\ell L\ell'}C^{\Theta\Theta}_{\ell'} + p_{\ell\ell'L}W^{x,0}_{\ell'L\ell}C^{\Theta\Theta}_{\ell}\,. \tag{85}$$

The sign $p_{\ell\ell''L}$ depends on the parity of W; $p_{\ell\ell''L}=1$ for the even parity fields (e.g. $x=\phi,\epsilon$) and -1 for the odd parity fields (e.g. $x=\varpi,\alpha$).

3.3.2 ΘE

In the ΘE estimator, the two contributions are given as

$$\begin{aligned}\langle\Theta_{\ell m}(\delta E_{\ell'm'})\rangle &= \sum_{LM\ell''m''}(-1)^{m'}\begin{pmatrix}\ell' & L & \ell''\\ -m' & M & m''\end{pmatrix}x_{LM}[\langle\Theta_{\ell m}E_{\ell''m''}\rangle W^{x,+}_{\ell'L\ell''} + \langle\Theta_{\ell m}B_{\ell''m''}\rangle W^{x,-}_{\ell'L\ell''}] \\ &= \sum_{LM}(-1)^{m+m'}\begin{pmatrix}\ell' & L & \ell\\ -m' & M & -m\end{pmatrix}x_{LM}[C^{\Theta\mathrm{E}}_{\ell}W^{x,+}_{\ell'L\ell} + C^{\Theta\mathrm{B}}_{\ell}W^{x,-}_{\ell'L\ell}] \\ &= \sum_{LM}p_{\ell L\ell'}\begin{pmatrix}\ell' & L & \ell\\ m' & M & m\end{pmatrix}x^*_{LM}[C^{\Theta\mathrm{E}}_{\ell}W^{x,+}_{\ell'L\ell} + C^{\Theta\mathrm{B}}_{\ell}W^{x,-}_{\ell'L\ell}] \\ &= \sum_{LM}p_{\ell L\ell'}\begin{pmatrix}\ell & \ell' & L\\ m & m' & M\end{pmatrix}x^*_{LM}[C^{\Theta\mathrm{E}}_{\ell}W^{x,+}_{\ell'L\ell} + C^{\Theta\mathrm{B}}_{\ell}W^{x,-}_{\ell'L\ell}]\,, \end{aligned} \tag{86}$$

and

$$\begin{aligned}\langle(\delta\Theta_{\ell m})E_{\ell'm'}\rangle &= \sum_{LM\ell''m''}(-1)^m\begin{pmatrix}\ell & L & \ell''\\ -m & M & m''\end{pmatrix}x_{LM}\langle E_{\ell'm'}\Theta_{\ell''m''}\rangle W^{x,0}_{\ell L\ell''} \\ &= \sum_{LM}(-1)^{m+m'}\begin{pmatrix}\ell & L & \ell'\\ -m & M & -m'\end{pmatrix}x_{LM}W^{x,0}_{\ell L\ell'}C^{\Theta\mathrm{E}}_{\ell'} \\ &= \sum_{LM}p_{\ell L\ell'}\begin{pmatrix}\ell & L & \ell'\\ m & M & m'\end{pmatrix}x^*_{LM}W^{x,0}_{\ell L\ell'}C^{\Theta\mathrm{E}}_{\ell'} \\ &= \sum_{LM}\begin{pmatrix}\ell & \ell' & L\\ m & m' & M\end{pmatrix}x^*_{LM}W^{x,0}_{\ell L\ell'}C^{\Theta\mathrm{E}}_{\ell'}\,. \end{aligned} \tag{87}$$

If we decompose the terms into the following two parts,

$$\begin{aligned}
f^{x,(\Theta E),+}_{\ell L\ell'} &= W^{x,0}_{\ell L\ell'}C^{\Theta\mathrm{E}}_{\ell'} + p_{\ell L\ell'}W^{x,+}_{\ell' L\ell}C^{\Theta\mathrm{E}}_{\ell}\,, \\
f^{x,(\Theta E),-}_{\ell L\ell'} &= p_{\ell L\ell'}C^{\Theta\mathrm{B}}_{\ell}W^{x,-}_{\ell' L\ell}\,,
\end{aligned} \tag{88}$$

the above two parts are orthogonal each other.

3.3.3 ΘB

In the ΘB estimator, the two contributions are given as

$$\begin{aligned}
\langle\Theta_{\ell m}\delta B_{\ell' m'}\rangle &= \sum_{LM\ell''m''}(-1)^{m'}\begin{pmatrix}\ell' & L & \ell'' \\ -m' & M & m''\end{pmatrix}x_{LM}[-\langle\Theta_{\ell m}E_{\ell''m''}\rangle W^{x,-}_{\ell' L\ell''} + \langle\Theta_{\ell m}B_{\ell''m''}\rangle W^{x,+}_{\ell' L\ell''}] \\
&= \sum_{LM}(-1)^{m+m'}\begin{pmatrix}\ell' & L & \ell \\ -m' & M & -m\end{pmatrix}x_{LM}[-C^{\Theta\mathrm{E}}_{\ell}W^{x,-}_{\ell' L\ell} + C^{\Theta\mathrm{B}}_{\ell}W^{x,+}_{\ell' L\ell}] \\
&= -\sum_{LM}p_{\ell L\ell'}\begin{pmatrix}\ell' & L & \ell \\ m' & M & m\end{pmatrix}x^*_{LM}[C^{\Theta\mathrm{E}}_{\ell}W^{x,-}_{\ell' L\ell} - C^{\Theta\mathrm{B}}_{\ell}W^{x,+}_{\ell' L\ell}] \\
&= -\sum_{LM}p_{\ell L\ell'}\begin{pmatrix}\ell & \ell' & L \\ m & m' & M\end{pmatrix}x^*_{LM}[C^{\Theta\mathrm{E}}_{\ell}W^{x,-}_{\ell' L\ell} - C^{\Theta\mathrm{B}}_{\ell}W^{x,+}_{\ell' L\ell}]\,,
\end{aligned} \tag{89}$$

and

$$\begin{aligned}
\langle(\delta\Theta_{\ell m})B_{\ell' m'}\rangle &= \sum_{LM\ell''m''}(-1)^{m}\begin{pmatrix}\ell & L & \ell'' \\ -m & M & m''\end{pmatrix}x_{LM}\langle B_{\ell' m'}\Theta_{\ell''m''}\rangle W^{x,0}_{\ell L\ell''} \\
&= \sum_{LM}(-1)^{m+m'}\begin{pmatrix}\ell & L & \ell' \\ -m & M & -m'\end{pmatrix}x_{LM}W^{x,0}_{\ell L\ell'}C^{\Theta\mathrm{B}}_{\ell'} \\
&= \sum_{LM}p_{\ell L\ell'}\begin{pmatrix}\ell & L & \ell' \\ m & M & m'\end{pmatrix}x^*_{LM}W^{x,0}_{\ell L\ell'}C^{\Theta\mathrm{B}}_{\ell'} \\
&= \sum_{LM}\begin{pmatrix}\ell & \ell' & L \\ m & m' & M\end{pmatrix}x^*_{LM}W^{x,0}_{\ell L\ell'}C^{\Theta\mathrm{B}}_{\ell'}\,.
\end{aligned} \tag{90}$$

Combining the above two terms, we find

$$f^{x,(\Theta B)}_{\ell L\ell'} = W^{x,0}_{\ell L\ell'}C^{\Theta\mathrm{B}}_{\ell'} - p_{\ell L\ell'}[W^{x,-}_{\ell' L\ell}C^{\Theta\mathrm{E}}_{\ell} - W^{x,+}_{\ell' L\ell}C^{\Theta\mathrm{B}}_{\ell}]\,. \tag{91}$$

If we decompose the terms into the following two parts,

$$\begin{aligned}
f^{x,(\Theta B),+}_{\ell L\ell'} &= -p_{\ell L\ell'}W^{x,-}_{\ell' L\ell}C^{\Theta\mathrm{E}}_{\ell}\,, \\
f^{x,(\Theta B),-}_{\ell L\ell'} &= W^{x,0}_{\ell L\ell'}C^{\Theta\mathrm{B}}_{\ell'} + p_{\ell L\ell'}W^{x,+}_{\ell' L\ell}C^{\Theta\mathrm{B}}_{\ell}\,,
\end{aligned} \tag{92}$$

the above two parts are orthogonal each other.

3.3.4 EB

In the EB estimator, the two contributions are given as

$$\begin{aligned}\langle E_{\ell m}\delta B_{\ell' m'}\rangle &= \sum_{LM\ell'' m''}(-1)^{m'}\begin{pmatrix}\ell' & L & \ell''\\ -m' & M & m''\end{pmatrix}x_{LM}[-\langle E_{\ell m}E_{\ell'' m''}\rangle W^{x,-}_{\ell' L\ell''}+\langle E_{\ell m}B_{\ell'' m''}\rangle W^{x,+}_{\ell' L\ell''}]\\ &= \sum_{LM}(-1)^{m+m'}\begin{pmatrix}\ell' & L & \ell\\ -m' & M & -m\end{pmatrix}x_{LM}[-C^{\rm EE}_{\ell}W^{x,-}_{\ell' L\ell}+C^{\rm EB}_{\ell}W^{x,+}_{\ell' L\ell}]\\ &= -\sum_{LM}p_{\ell L\ell'}\begin{pmatrix}\ell' & L & \ell\\ m' & M & m\end{pmatrix}x^*_{LM}[C^{\rm EE}_{\ell}W^{x,-}_{\ell' L\ell}-C^{\rm EB}_{\ell}W^{x,+}_{\ell' L\ell}]\\ &= -\sum_{LM}p_{\ell L\ell'}\begin{pmatrix}\ell & \ell' & L\\ m & m' & M\end{pmatrix}x^*_{LM}[C^{\rm EE}_{\ell}W^{x,-}_{\ell' L\ell}-C^{\rm EB}_{\ell}W^{x,+}_{\ell' L\ell}]\,,\end{aligned}\tag{93}$$

and

$$\begin{aligned}\langle(\delta E_{\ell m})B_{\ell' m'}\rangle &= \sum_{LM\ell'' m''}(-1)^{m}\begin{pmatrix}\ell & L & \ell''\\ -m & M & m''\end{pmatrix}x_{LM}[\langle B_{\ell' m'}E_{\ell'' m''}\rangle W^{x,+}_{\ell L\ell''}+\langle B_{\ell' m'}B_{\ell'' m''}\rangle W^{x,-}_{\ell L\ell''}]\\ &= \sum_{LM}(-1)^{m+m'}\begin{pmatrix}\ell & L & \ell'\\ -m & M & -m'\end{pmatrix}x_{LM}[C^{\rm EB}_{\ell'}W^{x,+}_{\ell L\ell'}+C^{\rm BB}_{\ell'}W^{x,-}_{\ell L\ell'}]\\ &= \sum_{LM}\begin{pmatrix}\ell & \ell' & L\\ m & m' & M\end{pmatrix}x^*_{LM}[C^{\rm EB}_{\ell'}W^{x,+}_{\ell L\ell'}+C^{\rm BB}_{\ell'}W^{x,-}_{\ell L\ell'}]\,.\end{aligned}\tag{94}$$

Combining the above two terms, we find

$$f^{x,(EB)}_{\ell L\ell'} = C^{\rm EB}_{\ell'}W^{x,+}_{\ell L\ell'}+C^{\rm BB}_{\ell'}W^{x,-}_{\ell L\ell'}-p_{\ell L\ell'}[C^{\rm EE}_{\ell}W^{x,-}_{\ell' L\ell}-C^{\rm EB}_{\ell}W^{x,+}_{\ell' L\ell}]\,.\tag{95}$$

If we decompose the terms into the following two parts,

$$\begin{aligned}f^{x,(EB),+}_{\ell L\ell'} &= C^{\rm BB}_{\ell'}W^{x,-}_{\ell L\ell'}-p_{\ell L\ell'}C^{\rm EE}_{\ell}W^{x,-}_{\ell' L\ell}\,,\\ f^{x,(EB),-}_{\ell L\ell'} &= C^{\rm EB}_{\ell'}W^{x,+}_{\ell L\ell'}+p_{\ell L\ell'}C^{\rm EB}_{\ell}W^{x,+}_{\ell' L\ell}\,,\end{aligned}\tag{96}$$

the above two parts are orthogonal each other. This indicates that, if $C^{\rm EB}_{\ell}$ is non-zero due to the global rotation, even parity fields (lensing, window) leak into the odd parity estimator (rotation, curl mode) and introduce a mean-field;

$$\langle\widehat{\alpha}_{LM}\rangle = \alpha_{LM}+A^{\alpha,EB}_{L}\sum_{x=\phi,\tau,\dots}x_{LM}\frac{1}{2L+1}\sum_{\ell\ell'}g^{\alpha,EB}_{\ell L\ell'}f^{x,EB,{\rm even}}_{\ell L\ell'}\,.\tag{97}$$

3.3.5 EE

In the EE estimator, the two contributions are given as

$$\begin{aligned}\langle(\delta E_{\ell m})E_{\ell' m'}\rangle &= \sum_{LM\ell'' m''}(-1)^{m}\begin{pmatrix}\ell & L & \ell''\\ -m & M & m''\end{pmatrix}x_{LM}[\langle E_{\ell' m'}E_{\ell'' m''}\rangle W^{x,+}_{\ell L\ell''}+\langle E_{\ell' m'}B_{\ell'' m''}\rangle W^{x,-}_{\ell L\ell''}]\\ &= \sum_{LM}(-1)^{m+m'}\begin{pmatrix}\ell & L & \ell'\\ -m & M & -m'\end{pmatrix}x_{LM}[C^{\rm EE}_{\ell'}W^{x,+}_{\ell L\ell'}+C^{\rm EB}_{\ell'}W^{x,-}_{\ell L\ell'}]\\ &= \sum_{LM}\begin{pmatrix}\ell & \ell' & L\\ m & m' & M\end{pmatrix}x^*_{LM}[C^{\rm EE}_{\ell'}W^{x,+}_{\ell L\ell'}+C^{\rm EB}_{\ell'}W^{x,-}_{\ell L\ell'}]\,,\end{aligned}\tag{98}$$

and

$$\begin{aligned}\langle E_{\ell m}(\delta E_{\ell' m'})\rangle &= \sum_{LM}\begin{pmatrix}\ell' & \ell & L\\ m' & m & M\end{pmatrix}x^*_{LM}[C^{\rm EE}_{\ell'}W^{x,+}_{\ell L\ell'}+C^{\rm EB}_{\ell'}W^{x,-}_{\ell L\ell'}]\\ &= \sum_{LM}p_{\ell\ell' L}\begin{pmatrix}\ell & \ell' & L\\ m & m' & M\end{pmatrix}x^*_{LM}[C^{\rm EE}_{\ell'}W^{x,+}_{\ell L\ell'}+C^{\rm EB}_{\ell'}W^{x,-}_{\ell L\ell'}]\,.\end{aligned}\tag{99}$$

Combining the above two terms, we find

$$f^{x,(EE)}_{\ell L\ell'} = C^{\rm EE}_{\ell'}W^{x,+}_{\ell L\ell'} + C^{\rm EB}_{\ell'}W^{x,-}_{\ell L\ell'} + p_{\ell L\ell'}[C^{\rm EE}_{\ell}W^{x,+}_{\ell' L\ell} + C^{\rm EB}_{\ell}W^{x,-}_{\ell' L\ell}]\,. \tag{100}$$

If we decompose the terms into the following two parts,

$$\begin{aligned} f^{x,(EE),+}_{\ell L\ell'} &= C^{\rm EE}_{\ell'}W^{x,+}_{\ell L\ell'} + p_{\ell L\ell'}C^{\rm EE}_{\ell}W^{x,+}_{\ell' L\ell}\,, \\ f^{x,(EE),-}_{\ell L\ell'} &= C^{\rm EB}_{\ell'}W^{x,-}_{\ell L\ell'} + p_{\ell L\ell'}C^{\rm EB}_{\ell}W^{x,-}_{\ell' L\ell}\,, \end{aligned} \tag{101}$$

the above two parts are orthogonal each other.

3.3.6 BB

In the BB estimator, the two contributions are given as

$$\begin{aligned} \langle B_{\ell m}\delta B_{\ell' m'}\rangle &= \sum_{LM\ell'' m''}(-1)^{m'}\begin{pmatrix}\ell' & L & \ell'' \\ -m' & M & m''\end{pmatrix} x_{LM}[-\langle B_{\ell m}E_{\ell'' m''}\rangle W^{x,-}_{\ell' L\ell''} + \langle B_{\ell m}B_{\ell'' m''}\rangle W^{x,+}_{\ell' L\ell''}] \\ &= \sum_{LM}(-1)^{m+m'}\begin{pmatrix}\ell' & L & \ell \\ -m' & M & -m\end{pmatrix} x_{LM}[-C^{\rm EB}_{\ell}W^{x,-}_{\ell' L\ell} + C^{\rm BB}_{\ell}W^{x,+}_{\ell' L\ell}] \\ &= \sum_{LM} p_{\ell L\ell'}\begin{pmatrix}\ell' & L & \ell \\ m' & M & m\end{pmatrix} x^*_{LM}[-C^{\rm EB}_{\ell}W^{x,-}_{\ell' L\ell} + C^{\rm BB}_{\ell}W^{x,+}_{\ell' L\ell}] \\ &= \sum_{LM} p_{\ell L\ell'}\begin{pmatrix}\ell & \ell' & L \\ m & m' & M\end{pmatrix} x^*_{LM}[-C^{\rm EB}_{\ell}W^{x,-}_{\ell' L\ell} + C^{\rm BB}_{\ell}W^{x,+}_{\ell' L\ell}]\,, \end{aligned} \tag{102}$$

and, by exchanging (ℓ, m) and (ℓ', m') in the above equation:

$$\begin{aligned} \langle(\delta B_{\ell m})B_{\ell' m'}\rangle &= \sum_{LM} p_{\ell L\ell'}\begin{pmatrix}\ell' & \ell & L \\ m' & m & M\end{pmatrix} x^*_{LM}[-C^{\rm EB}_{\ell'}W^{x,-}_{\ell L\ell'} + C^{\rm BB}_{\ell'}W^{x,+}_{\ell L\ell'}] \\ &= \sum_{LM}\begin{pmatrix}\ell & \ell' & L \\ m & m' & M\end{pmatrix} x^*_{LM}[-C^{\rm EB}_{\ell'}W^{x,-}_{\ell L\ell'} + C^{\rm BB}_{\ell'}W^{x,+}_{\ell L\ell'}]\,. \end{aligned} \tag{103}$$

Combining the above two terms, we find

$$f^{x,(BB)}_{\ell L\ell'} = p_{\ell L\ell'}[-C^{\rm EB}_{\ell}W^{x,-}_{\ell' L\ell} + C^{\rm BB}_{\ell}W^{x,+}_{\ell' L\ell}] - C^{\rm EB}_{\ell'}W^{x,-}_{\ell L\ell'} + C^{\rm BB}_{\ell'}W^{x,+}_{\ell L\ell'}\,. \tag{104}$$

If we decompose the terms into the following two parts,

$$\begin{aligned} f^{x,(BB),+}_{\ell L\ell'} &= C^{\rm BB}_{\ell'}W^{x,+}_{\ell L\ell'} + p_{\ell L\ell'}C^{\rm BB}_{\ell}W^{x,+}_{\ell' L\ell} \\ f^{x,(BB),-}_{\ell L\ell'} &= -C^{\rm EB}_{\ell'}W^{x,-}_{\ell L\ell'} - p_{\ell L\ell'}C^{\rm EB}_{\ell}W^{x,-}_{\ell' L\ell} = -f^{x,(EE),-}_{\ell L\ell'}\,, \end{aligned} \tag{105}$$

the above two parts are orthogonal each other.

3.4 Additive distortions

Point or extended-sources and inhomogeneous noise can also produce mode couplings. For circular sources, we assume that the fields are given by

$$s^i(\hat{\boldsymbol{n}}) = f^i\theta(R^i\hat{\boldsymbol{n}}) = \sum_{\ell} f^i b_\ell Y_{\ell 0}(R^i\hat{\boldsymbol{n}}) = \sum_{\ell} f^i b_\ell \sum_{m'} D^{\ell}_{m'0}(R^i)Y_{\ell m'}(\hat{\boldsymbol{n}}) = \sum_{\ell m'} f^i y^i_{\ell m'}Y_{\ell m'}(\hat{\boldsymbol{n}})\,. \tag{106}$$

Using, $D^{\ell}_{m0}(\hat{\boldsymbol{n}}_i)=(4\pi/(2\ell+1))^{1/2}Y^*_{\ell m}(\hat{\boldsymbol{n}}_i)$, the additive anisotropies in the temperature quadratic estimator are given by

$$\begin{aligned}\langle s^i_{\ell m}s^j_{\ell' m'}\rangle &= \langle f_i^2\rangle\delta_{ij}y^i_{\ell m}y^i_{\ell' m'}\\ &= \langle f_i^2\rangle\delta_{ij}b_\ell b_{\ell'}[Y_{\ell m}(\hat{\boldsymbol{n}}_i)Y_{\ell' m'}(\hat{\boldsymbol{n}}_i)]^*\\ &= \langle f_i^2\rangle\delta_{ij}b_\ell b_{\ell'}\sum_{LM}\gamma_{\ell\ell' L}Y_{LM}(\hat{\boldsymbol{n}}_i)\begin{pmatrix}\ell & \ell' & L\\ 0 & 0 & 0\end{pmatrix}\begin{pmatrix}\ell & \ell' & L\\ m & m' & M\end{pmatrix}\\ &\equiv \sum_{LM}\begin{pmatrix}\ell & \ell' & L\\ m & m' & M\end{pmatrix}f^{s,(\Theta\Theta)}_{\ell L\ell'}\sigma^*_{i,LM}\,, \end{aligned}\tag{107}$$

where

$$\sigma_{i,LM}=f_i^2Y^*_{LM}(\hat{\boldsymbol{n}}_i)\tag{108}$$

$$f^{s,(\Theta\Theta)}_{\ell L\ell'}=b_\ell b_{\ell'}\gamma_{\ell L\ell'}\begin{pmatrix}\ell & L & \ell'\\ 0 & 0 & 0\end{pmatrix}=b_\ell b_{\ell'}W^{\epsilon,0}_{\ell L\ell'}\,.\tag{109}$$

Thus, we obtain the following relation:

$$f^{s,(\Theta\Theta)}_{\ell L\ell'}=b_\ell b_{\ell'}f^{\epsilon,(\Theta\Theta)}_{\ell L\ell'}|_{C^{\Theta\Theta}_\ell=1/2}\,.\tag{110}$$

We define

$$W^{s,(\Theta\Theta)}_{\ell L\ell'}\equiv b_\ell b_{\ell'}\frac{W^{\epsilon,(\Theta\Theta)}_{\ell L\ell'}}{2C^{\Theta\Theta}_{\ell'}}\,.\tag{111}$$

Alternatively, if $\langle s(\hat{\boldsymbol{n}})s(\hat{\boldsymbol{n}}')\rangle\propto\delta(\hat{\boldsymbol{n}}-\hat{\boldsymbol{n}}')$,

$$\begin{aligned}\langle s_{\ell m}s_{\ell' m'}\rangle &= \int \mathrm{d}^2\hat{\boldsymbol{n}}\int\mathrm{d}^2\hat{\boldsymbol{n}}'\;Y^*_{\ell m}(\hat{\boldsymbol{n}})Y^*_{\ell' m'}(\hat{\boldsymbol{n}}')\langle s^i(\hat{\boldsymbol{n}})s^i(\hat{\boldsymbol{n}}')\rangle\\ &= \int \mathrm{d}^2\hat{\boldsymbol{n}}\;Y^*_{\ell m}(\hat{\boldsymbol{n}})Y^*_{\ell' m'}(\hat{\boldsymbol{n}})\langle\sigma(\hat{\boldsymbol{n}})\rangle\\ &= \int \mathrm{d}^2\hat{\boldsymbol{n}}\;Y^*_{\ell m}(\hat{\boldsymbol{n}})Y^*_{\ell' m'}(\hat{\boldsymbol{n}})\sum_{LM}\sigma_{LM}Y_{LM}(\hat{\boldsymbol{n}})\\ &= \sum_{LM}\sigma_{LM}\int\mathrm{d}^2\hat{\boldsymbol{n}}\;(-1)^{m+m'}Y_{\ell,-m}(\hat{\boldsymbol{n}})Y_{\ell',-m'}(\hat{\boldsymbol{n}})Y_{LM}(\hat{\boldsymbol{n}})\\ &= \sum_{LM}\begin{pmatrix}\ell & \ell' & L\\ m & m' & M\end{pmatrix}f^{s,(\Theta\Theta)}_{\ell L\ell'}\sigma^*_{LM}\,,\end{aligned}\tag{112}$$

with $b_\ell=1$.

3.5 Summary

The weight functions are given as [2]

$$f^{x,(\Theta\Theta)}_{\ell L\ell'}=W^{x,0}_{\ell L\ell'}C^{\Theta\Theta}_{\ell'}+p_{\ell L\ell'}W^{x,0}_{\ell' L\ell}C^{\Theta\Theta}_{\ell}\,,\tag{113}$$

$$f^{x,(\Theta E)}_{\ell L\ell'}=W^{x,0}_{\ell L\ell'}C^{\Theta\mathrm{E}}_{\ell'}+p_{\ell L\ell'}W^{x,+}_{\ell' L\ell}C^{\Theta\mathrm{E}}_{\ell}\,,\tag{114}$$

$$f^{x,(\Theta B)}_{\ell L\ell'}=-p_{\ell L\ell'}W^{x,-}_{\ell' L\ell}C^{\Theta\mathrm{E}}_{\ell}\,,\tag{115}$$

$$f^{x,(EE)}_{\ell L\ell'}=W^{x,+}_{\ell L\ell'}C^{\mathrm{EE}}_{\ell'}+p_{\ell L\ell'}W^{x,+}_{\ell' L\ell}C^{\mathrm{EE}}_{\ell}\,,\tag{116}$$

$$f^{x,(EB)}_{\ell L\ell'}=W^{x,-}_{\ell L\ell'}C^{\mathrm{BB}}_{\ell'}-p_{\ell L\ell'}W^{x,-}_{\ell' L\ell}C^{\mathrm{EE}}_{\ell}\,,\tag{117}$$

$$f^{x,(BB)}_{\ell L\ell'}=W^{x,+}_{\ell L\ell'}C^{\mathrm{BB}}_{\ell'}+p_{\ell L\ell'}W^{x,+}_{\ell' L\ell}C^{\mathrm{BB}}_{\ell}\,.\tag{118}$$

[2] The original paper [12] has an opposite sign in front of the BB spectrum in EB estimator.

Note that the above weight functions are consistent with Ref. [9] ($W^{x,-}_{\ell L\ell'} = -_{\ominus}S^{x}_{\ell L\ell'}$) for the lensing case. In addition, the weight functions due to the presence of ΘB and EB are given by

$$f^{x,(\Theta E)}_{\ell L\ell'} = p_{\ell L\ell'} C^{\Theta\mathrm{B}}_{\ell} W^{x,-}_{\ell' L\ell} \,, \tag{119}$$
$$f^{x,(\Theta B)}_{\ell L\ell'} = W^{x,0}_{\ell L\ell'} C^{\Theta\mathrm{B}}_{\ell'} + p_{\ell L\ell'} W^{x,+}_{\ell' L\ell} C^{\Theta\mathrm{B}}_{\ell} \,, \tag{120}$$
$$f^{x,(EE)}_{\ell L\ell'} = W^{x,-}_{\ell L\ell'} C^{\mathrm{EB}}_{\ell'} + p_{\ell L\ell'} W^{x,-}_{\ell' L\ell} C^{\mathrm{EB}}_{\ell} \,, \tag{121}$$
$$f^{x,(EB)}_{\ell L\ell'} = W^{x,+}_{\ell L\ell'} C^{\mathrm{EB}}_{\ell'} + p_{\ell L\ell'} W^{x,+}_{\ell' L\ell} C^{\mathrm{EB}}_{\ell} \,, \tag{122}$$
$$f^{x,(BB)}_{\ell L\ell'} = -f^{x,(EE)}_{\ell L\ell'} \,. \tag{123}$$

It is also convenient to introduce a parity indicator:

$$p_{\phi} = 1 \,, \quad p_{\varpi} = -1 \,, \quad p_{\epsilon} = 1 \,, \quad p_{s} = 1 \,, \quad p_{\alpha} = -1 \,. \tag{124}$$

Then, we can replace $p_{\ell L\ell'}$ with p_x:

$$p_{\ell L\ell'} W^{x,0}_{\ell' L\ell} = p_x W^{x,0}_{\ell' L\ell} \,, \tag{125}$$
$$p_{\ell L\ell'} W^{x,+}_{\ell' L\ell} = p_x W^{x,+}_{\ell' L\ell} \,, \tag{126}$$
$$p_{\ell L\ell'} W^{x,-}_{\ell' L\ell} = -p_x W^{x,-}_{\ell' L\ell} \,. \tag{127}$$

The weight functions are given by

$$f^{x,(\Theta\Theta)}_{\ell L\ell'} = W^{x,0}_{\ell L\ell'} C^{\Theta\Theta}_{\ell'} + p_x W^{x,0}_{\ell' L\ell} C^{\Theta\Theta}_{\ell} \,, \tag{128}$$
$$f^{x,(\Theta E)}_{\ell L\ell'} = W^{x,0}_{\ell L\ell'} C^{\Theta\mathrm{E}}_{\ell'} + p_x W^{x,+}_{\ell' L\ell} C^{\Theta\mathrm{E}}_{\ell} \,, \tag{129}$$
$$f^{x,(\Theta B)}_{\ell L\ell'} = p_x W^{x,-}_{\ell' L\ell} C^{\Theta\mathrm{E}}_{\ell} \,, \tag{130}$$
$$f^{x,(EE)}_{\ell L\ell'} = W^{x,+}_{\ell L\ell'} C^{\mathrm{EE}}_{\ell'} + p_x W^{x,+}_{\ell' L\ell} C^{\mathrm{EE}}_{\ell} \,, \tag{131}$$
$$f^{x,(EB)}_{\ell L\ell'} = W^{x,-}_{\ell L\ell'} C^{\mathrm{BB}}_{\ell'} + p_x W^{x,-}_{\ell' L\ell} C^{\mathrm{EE}}_{\ell} \,, \tag{132}$$
$$f^{x,(BB)}_{\ell L\ell'} = W^{x,+}_{\ell L\ell'} C^{\mathrm{BB}}_{\ell'} + p_x W^{x,+}_{\ell' L\ell} C^{\mathrm{BB}}_{\ell} \,, \tag{133}$$

and [13]

$$f^{x,(\Theta E)}_{\ell L\ell'} = -p_x C^{\Theta\mathrm{B}}_{\ell} W^{x,-}_{\ell' L\ell} \,, \tag{134}$$
$$f^{x,(\Theta B)}_{\ell L\ell'} = W^{x,0}_{\ell L\ell'} C^{\Theta\mathrm{B}}_{\ell'} + p_x W^{x,+}_{\ell' L\ell} C^{\Theta\mathrm{B}}_{\ell} \,, \tag{135}$$
$$f^{x,(EE)}_{\ell L\ell'} = W^{x,-}_{\ell L\ell'} C^{\mathrm{EB}}_{\ell'} - p_x W^{x,-}_{\ell' L\ell} C^{\mathrm{EB}}_{\ell} \,, \tag{136}$$
$$f^{x,(EB)}_{\ell L\ell'} = W^{x,+}_{\ell L\ell'} C^{\mathrm{EB}}_{\ell'} + p_x W^{x,+}_{\ell' L\ell} C^{\mathrm{EB}}_{\ell} \,, \tag{137}$$
$$f^{x,(BB)}_{\ell L\ell'} = -f^{x,(EE)}_{\ell L\ell'} \,. \tag{138}$$

4 Computing quadratic estimator

4.1 Spherical Harmonics

The polarization vectors satisfy, $\boldsymbol{e}\cdot\boldsymbol{e}^*=1$, and, $\boldsymbol{e}\cdot\boldsymbol{e}=\boldsymbol{e}^*\cdot\boldsymbol{e}^*=0$. We obtain

$$-\boldsymbol{\nabla}Y^s_{\ell m}=\sqrt{\frac{(\ell-s)(\ell+s+1)}{2}}Y^{s+1}_{\ell m}\boldsymbol{e}^*-\sqrt{\frac{(\ell+s)(\ell-s+1)}{2}}Y^{s-1}_{\ell m}\boldsymbol{e}\,. \tag{139}$$

The complex conjugate is $(Y^s_{\ell m})^*=(-1)^{s+m}Y^{-s}_{\ell,-m}$. In particular, for $s=0$,

$$-\boldsymbol{\nabla}Y^*_{\ell m}=\sqrt{\frac{\ell(\ell+1)}{2}}\left((Y^1_{\ell m})^*\boldsymbol{e}-(Y^{-1}_{\ell m})^*\boldsymbol{e}^*\right), \tag{140}$$

and, for $s=-2$,

$$\begin{aligned}-\boldsymbol{\nabla}Y^{-2}_{\ell m}&=\sqrt{\frac{(\ell+2)(\ell-1)}{2}}Y^{-1}_{\ell m}\boldsymbol{e}^*-\sqrt{\frac{(\ell-2)(\ell+3)}{2}}Y^{-3}_{\ell m}\boldsymbol{e}\,,\\ -\boldsymbol{\nabla}(Y^{-2}_{\ell m})^*&=\sqrt{\frac{(\ell+2)(\ell-1)}{2}}(Y^{-1}_{\ell m})^*\boldsymbol{e}-\sqrt{\frac{(\ell-2)(\ell+3)}{2}}(Y^{-3}_{\ell m})^*\boldsymbol{e}^*\,.\end{aligned} \tag{141}$$

4.2 Healpix

`Healpix` is a useful public package for fullsky analysis [14]. Here, we consider the `Healpix` spin-s harmonic transform of a map $S(\hat{\boldsymbol{n}})=S^+(\hat{\boldsymbol{n}})+\mathrm{i}S^-(\hat{\boldsymbol{n}})$ where $S^\pm$ is real and $s\geq 0$. The harmonic coefficient is given by

$$S^++\mathrm{i}S^-=\sum_{\ell m}a^s_{\ell m}Y^s_{\ell m}\,. \tag{142}$$

Note that $a^{-s}_{\ell m}$ is defined as

$$S^+-\mathrm{i}S^-=\sum_{\ell m}a^{-s}_{\ell m}Y^{-s}_{\ell m}\,. \tag{143}$$

Then we obtain $(a^s_{\ell m})^*=(-1)^{m+s}a^{-s}_{\ell,-m}$. The subroutine `map2alm_spin` transform $S^\pm$ to $a^{s,\pm}_{\ell m}$ where

$$a^{s,+}_{\ell m}=-\frac{a^s_{\ell m}+(-1)^sa^{-s}_{\ell m}}{2} \tag{144}$$

$$a^{s,-}_{\ell m}=-\frac{a^s_{\ell m}-(-1)^sa^{-s}_{\ell m}}{2\mathrm{i}}\,, \tag{145}$$

are the rotational invariant coefficients with parity even and odd, respectively. Note that, identifying $S^+=Q$, $S^-=U$, $a^{2,+}_{lm}=E_{\ell m}$ and $a^{2,-}_{\ell m}=B_{\ell m}$, we obtain

$$Q+\mathrm{i}U=-\sum_{\ell m}(E_{\ell m}+\mathrm{i}B_{\ell m})Y^2_{\ell m}\,. \tag{146}$$

Since $(a^s_{\ell m})^*=(-1)^{m+s}a^{-s}_{\ell,-m}$, the above coefficients satisfy

$$(a^{s,\pm}_{\ell m})^*=(-1)^ma^{s,\pm}_{\ell,-m}\,. \tag{147}$$

On the other hand, `alm2map_spin` transform $a^{s,\pm}_{\ell m}$ to $S^\pm$, but $a^{s,\pm}_{\ell m}$ should satisfy the above condition. Note that, with $S\equiv S^++\mathrm{i}S^-$, we find

$$a^{s,+}_{\ell m}=-\frac{1}{2}\int\mathrm{d}\hat{\boldsymbol{n}}\;[(Y^s_{\ell m})^*S+(-1)^s(Y^{-s}_{\ell m})^*S^*]\,, \tag{148}$$

$$a^{s,-}_{\ell m}=-\frac{1}{2\mathrm{i}}\int\mathrm{d}\hat{\boldsymbol{n}}\;[(Y^s_{\ell m})^*S-(-1)^s(Y^{-s}_{\ell m})^*S^*]\,. \tag{149}$$

Let us consider the case we want to transform $a_{\ell m}$ with a spin-s spherical harmonics using `alm2map_spin`. The outputs, $S^{\pm}$, are given by:

$$S^{+} + \mathrm{i}S^{-} = \sum_{\ell m} a_{\ell m} Y^{s}_{\ell m} \,. \tag{150}$$

The complex conjugate of the above quantity becomes

$$S^{+} - \mathrm{i}S^{-} = (-1)^{s} \sum_{\ell m} a_{\ell m} Y^{-s}_{\ell m} \,. \tag{151}$$

The inputs of `alm2map_spin` become

$$a^{s,+}_{\ell m} = -a_{\ell m} \,, \tag{152}$$
$$a^{s,-}_{\ell m} = 0 \,. \tag{153}$$

4.3 Lensing

Here, we focus on how to compute the unnormalized lensing estimators.

4.3.1 Convolution formula for lensing

For convenience, we define

$$\overline{X}^{s}_{Y}(\hat{\boldsymbol{n}}) = \sum_{\ell m} C^{XY}_{\ell} \overline{X}_{\ell m} Y^{s}_{\ell m}(\hat{\boldsymbol{n}}) \,, \tag{154}$$

with $\overline{X}_{\ell m} = \widehat{X}_{\ell m}/\widehat{C}^{XY}_{\ell}$ being the inverse-variance filtered multipoles. We also define the inverse-variance filtered temperature map and the Stokes Q/U map constructed from the inverse-variance filtered E or B alone:

$$\overline{\Theta} = \sum_{\ell m} \overline{\Theta}_{\ell m} Y_{\ell m} \,, \tag{155}$$
$$\overline{P}^{E} = \overline{Q}^{E} + \mathrm{i}\overline{U}^{E} \equiv - \sum_{\ell m} Y^{2}_{\ell m} \overline{E}_{\ell m} \,, \tag{156}$$
$$\overline{P}^{B} = \overline{Q}^{B} + \mathrm{i}\overline{U}^{B} \equiv - \sum_{\ell m} Y^{2}_{\ell m} \mathrm{i}\overline{B}_{\ell m} \,. \tag{157}$$

In full-sky, the unnormalized quadratic estimator of the gradient and curl modes are given by [12, 9]:

$$\overline{\phi}^{(\alpha)}_{\ell m} = \int \mathrm{d}^{2}\hat{\boldsymbol{n}} \; [\boldsymbol{\nabla} Y^{*}_{\ell m}(\hat{\boldsymbol{n}})] \cdot \boldsymbol{v}^{(\alpha)}(\hat{\boldsymbol{n}}) \,, \tag{158}$$
$$\overline{\varpi}^{(\alpha)}_{\ell m} = \int \mathrm{d}^{2}\hat{\boldsymbol{n}} \; [(\star\boldsymbol{\nabla}) Y^{*}_{\ell m}(\hat{\boldsymbol{n}})] \cdot \boldsymbol{v}^{(\alpha)}(\hat{\boldsymbol{n}}) \,, \tag{159}$$

where we define

$$\boldsymbol{v}^{\Theta\Theta}(\hat{\boldsymbol{n}}) = \overline{\Theta} \boldsymbol{\nabla} \overline{\Theta}^{0}_{\Theta} \,, \tag{160}$$
$$\boldsymbol{v}^{\Theta E}(\hat{\boldsymbol{n}}) = \Re(\overline{P}^{E} \boldsymbol{\nabla} \overline{T}^{-2}_{E}) + \overline{\Theta} \boldsymbol{\nabla} \overline{E}_{\Theta} \,, \tag{161}$$
$$\boldsymbol{v}^{\Theta B}(\hat{\boldsymbol{n}}) = \Re(\overline{P}^{B} \boldsymbol{\nabla} \Theta^{-2}_{E}) \,, \tag{162}$$
$$\boldsymbol{v}^{EE}(\hat{\boldsymbol{n}}) = \Re(\overline{P}^{E} \boldsymbol{\nabla} \overline{E}^{-2}_{E}) \,, \tag{163}$$
$$\boldsymbol{v}^{EB}(\hat{\boldsymbol{n}}) = \Re(\overline{P}^{B} \boldsymbol{\nabla} \overline{E}^{-2}_{E}) + \Re(\overline{P}^{E} \boldsymbol{\nabla} \mathrm{i}\overline{B}^{-2}_{iB}) \,, \tag{164}$$
$$\boldsymbol{v}^{BB}(\hat{\boldsymbol{n}}) = \Re(\overline{P}^{B} \boldsymbol{\nabla} \mathrm{i}\overline{B}^{-2}_{iB}) \,. \tag{165}$$

The quantity $\boldsymbol{v}^{\Theta E}$ gives the nearly optimal estimator [12].

In general, we can decompose the 2D vector, $\boldsymbol{v}^\alpha$, into

$$\boldsymbol{v}^\alpha = \frac{v^\alpha_- \boldsymbol{e} + v^\alpha_+ \boldsymbol{e}^*}{\sqrt{2}} \,. \tag{166}$$

Since $\boldsymbol{v}^\alpha$ is real, we find $(v^\alpha_-)^* = v^\alpha_+ \equiv v^\alpha$. Then we obtain

$$\overline{\phi}^{(\alpha)}_{\ell m} = -\frac{\sqrt{\ell(\ell+1)}}{2}\int \mathrm{d}^2\hat{\boldsymbol{n}}\; [(Y^1_{\ell m})^* v^\alpha - (Y^{-1}_{\ell m})^*(v^\alpha)^*] = \sqrt{\ell(\ell+1)} v^{1,+}_{\ell m} \,, \tag{167}$$

$$\overline{\varpi}^{(\alpha)}_{\ell m} = -\frac{\sqrt{\ell(\ell+1)}}{2\mathrm{i}}\int \mathrm{d}^2\hat{\boldsymbol{n}}\; [(Y^1_{\ell m})^* v^\alpha + (Y^{-1}_{\ell m})^*(v^\alpha)^*] = \sqrt{\ell(\ell+1)} v^{1,-}_{\ell m} \,, \tag{168}$$

where $v^{1,\pm}_{\ell m}$ are the outputs of `map2alm_spin` by inputting, $S = v^\alpha$, with $s = 1$. Similarly, the imaginary lensing is expressed by replacing v with $\tilde{v}$. In the following subsections, we show v^α for each quadratic estimator.

4.3.2 Spin fields

We first define spin fields which are used for computing the estimators: The spin $0+1$ fields are

$$\Theta^+ + \mathrm{i}\Theta^- \equiv -\sum_{\ell m} \overline{\Theta}_{\ell m} C^{\Theta\Theta}_\ell \sqrt{\ell(\ell+1)} Y^1_{\ell m} = -\sum_{\ell m} \overline{\Theta}_{\ell m} C^{\Theta\Theta}_\ell \eth Y_{\ell m} \,, \tag{169}$$

$$E^+_1 + \mathrm{i}E^-_1 \equiv -\sum_{\ell m} \overline{E}_{\ell m} C^{\Theta\mathrm{E}}_\ell \sqrt{\ell(\ell+1)} Y^1_{\ell m} = -\sum_{\ell m} \overline{E}_{\ell m} C^{\Theta\mathrm{E}}_\ell \eth Y_{\ell m} \,. \tag{170}$$

The spin 2 ± 1 fields are

$$\begin{aligned}
\Theta^+_1 + \mathrm{i}\Theta^-_1 &\equiv -\sum_{\ell m} \overline{\Theta}_{\ell m} C^{\Theta\mathrm{E}}_\ell \sqrt{(\ell+2)(\ell-1)} Y^1_{\ell m} = \sum_{\ell m} \overline{\Theta}_{\ell m} C^{\Theta\mathrm{E}}_\ell \bar{\eth} Y^2_{\ell m} \,, \\
\Theta^+_3 + \mathrm{i}\Theta^-_3 &\equiv -\sum_{\ell m} \overline{\Theta}_{\ell m} C^{\Theta\mathrm{E}}_\ell \sqrt{(\ell-2)(\ell+3)} Y^3_{\ell m} = -\sum_{\ell m} \overline{\Theta}_{\ell m} C^{\Theta\mathrm{E}}_\ell \eth Y^2_{\ell m} \,, \\
\mathcal{E}^+_1 + \mathrm{i}\mathcal{E}^-_1 &\equiv -\sum_{\ell m} \overline{E}_{\ell m} C^{\mathrm{EE}}_\ell \sqrt{(\ell+2)(\ell-1)} Y^1_{\ell m} = \sum_{\ell m} \overline{E}_{\ell m} C^{\mathrm{EE}}_\ell \bar{\eth} Y^2_{\ell m} \,, \\
\mathcal{E}^+_3 + \mathrm{i}\mathcal{E}^-_3 &\equiv -\sum_{\ell m} \overline{E}_{\ell m} C^{\mathrm{EE}}_\ell \sqrt{(\ell-2)(\ell+3)} Y^3_{\ell m} = -\sum_{\ell m} \overline{E}_{\ell m} C^{\mathrm{EE}}_\ell \eth Y^2_{\ell m} \,, \\
\mathcal{B}^+_1 + \mathrm{i}\mathcal{B}^-_1 &\equiv -\sum_{\ell m} \mathrm{i}\overline{B}_{\ell m} C^{\mathrm{BB}}_\ell \sqrt{(\ell+2)(\ell-1)} Y^1_{\ell m} = \sum_{\ell m} \mathrm{i}\overline{B}_{\ell m} C^{\mathrm{BB}}_\ell \bar{\eth} Y^2_{\ell m} \,, \\
\mathcal{B}^+_3 + \mathrm{i}\mathcal{B}^-_3 &\equiv -\sum_{\ell m} \mathrm{i}\overline{B}_{\ell m} C^{\mathrm{BB}}_\ell \sqrt{(\ell-2)(\ell+3)} Y^3_{\ell m} = -\sum_{\ell m} \mathrm{i}\overline{B}_{\ell m} C^{\mathrm{BB}}_\ell \eth Y^2_{\ell m} \,.
\end{aligned} \tag{171}$$

4.3.3 $\Theta\Theta$

The estimator for $\Theta\Theta$ contains

$$\begin{aligned}
\boldsymbol{v}^{\Theta\Theta} &= \overline{\Theta}\sum_{\ell m} C^{\Theta\Theta}_\ell \overline{\Theta}_{\ell m} \boldsymbol{\nabla} Y_{\ell m} \\
&= \overline{\Theta}\sum_{\ell m} C^{\Theta\Theta}_\ell \overline{\Theta}_{\ell m} \sqrt{\frac{\ell(\ell+1)}{2}}\big(-Y^1_{\ell m}\boldsymbol{e}^* + Y^{-1}_{\ell m}\boldsymbol{e}\big) \\
&= \frac{1}{\sqrt{2}}\overline{\Theta}\big[(\Theta^+ + \mathrm{i}\Theta^-)\boldsymbol{e}^* + (\Theta^+ - \mathrm{i}\Theta^-)\boldsymbol{e}\big] \,.
\end{aligned} \tag{172}$$

We obtain

$$v^{\Theta\Theta} = \overline{\Theta}(\Theta^+ + \mathrm{i}\Theta^-) \,. \tag{173}$$

4.3.4 ΘE

The ΘE estimator contains;

$$\begin{aligned} \boldsymbol{v}^{\Theta E} &= \Re\left[(-\overline{Q}^E + \overline{U}^E)\sum_{\ell m} C_\ell^{\Theta E}\overline{\Theta}_{\ell m}\left(-\sqrt{\frac{(\ell+2)(\ell-1)}{2}}Y_{\ell m}^{-1}\boldsymbol{e}^* + \sqrt{\frac{(\ell-2)(\ell+3)}{2}}Y_{\ell m}^{-3}\boldsymbol{e}\right)\right] \\ &\quad + \overline{\Theta}\sum_{\ell m} C_\ell^{\Theta E}\overline{E}_{\ell m}\sqrt{\frac{\ell(\ell+1)}{2}}\left(-Y_{\ell m}^1\boldsymbol{e}^* + Y_{\ell m}^{-1}\boldsymbol{e}\right) \\ &= \frac{1}{2\sqrt{2}}\left[-(\overline{Q}^E + \mathrm{i}\overline{U}^E)[-(\Theta_1^+ - \mathrm{i}\Theta_1^-)\boldsymbol{e}^* + (\Theta_3^+ - \mathrm{i}\Theta_3^-)\boldsymbol{e}] + \mathrm{c.c.}\right] \\ &\quad + \frac{1}{\sqrt{2}}\overline{\Theta}[(E_1^+ + \mathrm{i}E_1^-)\boldsymbol{e}^* + (E_1^+ - \mathrm{i}E_1^-)\boldsymbol{e}]\,. \end{aligned} \tag{174}$$

The above quantities are obtained by `map2alm_spin`. We find that

$$\begin{aligned} v^{\Theta E} &= \frac{1}{2}[(\overline{Q}^E + \mathrm{i}\overline{U}^E)(\Theta_1^+ - \mathrm{i}\Theta_1^-) + (-\overline{Q}^E + \mathrm{i}\overline{U}^E)(\Theta_3^+ + \mathrm{i}\Theta_3^-)] + \overline{\Theta}(E_1^+ + \mathrm{i}E_1^-) \\ &= \frac{1}{2}[\overline{Q}^E(\Theta_1^+ - \Theta_3^+) + \overline{U}^E(\Theta_1^- - \Theta_3^-) + \mathrm{i}[-\overline{Q}^E(\Theta_1^- + \Theta_3^-) + \overline{U}^E(\Theta_1^+ + \Theta_3^+)]] + \overline{\Theta}(E_1^+ + \mathrm{i}E_1^-)\,. \end{aligned} \tag{175}$$

4.3.5 ΘB

The ΘB estimator is obtained by replacing E to $\mathrm{i}B$ in the ΘE estimator and ignore the second term;

$$v^{\Theta B} = \frac{1}{2}[\overline{Q}^B(\Theta_1^+ - \Theta_3^+) + \overline{U}^B(\Theta_1^- - \Theta_3^-) + \mathrm{i}[-\overline{Q}^B(\Theta_3^- + \Theta_1^-) + \overline{U}^B(\Theta_3^+ + \Theta_1^+)]]\,. \tag{176}$$

4.3.6 EE

The EE estimator contains;

$$\begin{aligned} \boldsymbol{v}^{EE} &= \frac{1}{2}(\overline{Q}^E + \mathrm{i}\overline{U}^E)\sum_{\ell m} C_\ell^{\mathrm{EE}}\overline{E}_{\ell m}\Big(-\sqrt{\frac{(\ell+2)(\ell-1)}{2}}Y_{\ell m}^{-1}\boldsymbol{e}^* + \sqrt{\frac{(\ell-2)(\ell+3)}{2}}Y_{\ell m}^{-3}\boldsymbol{e}\Big) + \mathrm{c.c.} \\ &= \frac{1}{2\sqrt{2}}(\overline{Q}^E + \mathrm{i}\overline{U}^E)[(\mathcal{E}_1^+ - \mathrm{i}\mathcal{E}_1^-)\boldsymbol{e}^* - (\mathcal{E}_3^+ - \mathrm{i}\mathcal{E}_3^-)\boldsymbol{e}] + \mathrm{c.c.}\,. \end{aligned} \tag{177}$$

Then we obtain

$$\begin{aligned} v^{EE} &= \frac{1}{2}(\overline{Q}^E + \mathrm{i}\overline{U}^E)[\mathcal{E}_1^+ - \mathrm{i}\mathcal{E}_1^-] + \frac{1}{2}(-\overline{Q}^E + \mathrm{i}\overline{U}^E)[\mathcal{E}_3^+ + \mathrm{i}\mathcal{E}_3^-] \\ &= \frac{1}{2}[\overline{Q}^E(\mathcal{E}_1^+ - \mathcal{E}_3^+) + \overline{U}^E(\mathcal{E}_1^- - \mathcal{E}_3^-)] + \frac{\mathrm{i}}{2}[-\overline{Q}^E(\mathcal{E}_3^- + \mathcal{E}_1^-) + \overline{U}^E(\mathcal{E}_3^+ + \mathcal{E}_1^+)]\,. \end{aligned} \tag{178}$$

4.3.7 BB

The BB estimator is the same as EE estimator but using B modes, and the result is;

$$v^{BB} = \frac{1}{2}[\overline{Q}^B(\mathcal{B}_1^+ - \mathcal{B}_3^+) + \overline{U}^B(\mathcal{B}_1^- - \mathcal{B}_3^-)] + \frac{\mathrm{i}}{2}[-\overline{Q}^B(\mathcal{B}_3^- + \mathcal{B}_1^-) + \overline{U}^B(\mathcal{B}_3^+ + \mathcal{B}_1^+)]\,. \tag{179}$$

4.3.8 EB

The first term of the EB estimator is obtained by replacing E to $\mathrm{i}B$ in the first half of the EE estimator. Similarly, the second term of the BB estimator is given by replacing $\mathrm{i}B$ to E in the first half of the BB estimator. The result is;

$$\begin{aligned} v^{EB} &= \frac{1}{2}[\overline{Q}^B(\mathcal{E}_1^+ - \mathcal{E}_3^+) + \overline{U}^B(\mathcal{E}_1^- - \mathcal{E}_3^-)] + \frac{\mathrm{i}}{2}[-\overline{Q}^B(\mathcal{E}_3^- + \mathcal{E}_1^-) + \overline{U}^B(\mathcal{E}_3^+ + \mathcal{E}_1^+)] \\ &\quad + \frac{1}{2}[\overline{Q}^E(\mathcal{B}_1^+ - \mathcal{B}_3^+) + \overline{U}^E(\mathcal{B}_1^- - \mathcal{B}_3^-)] + \frac{\mathrm{i}}{2}[-\overline{Q}^E(\mathcal{B}_3^- + \mathcal{B}_1^-) + \overline{U}^E(\mathcal{B}_3^+ + \mathcal{B}_1^+)]\,. \end{aligned} \tag{180}$$

4.4 Odd parity lensing

The estimator is given by

$$\overline{x}^{(XY)}_{\ell m} = \frac{1}{\Delta^{\rm XY}}\sum_{\ell m}\sum_{\ell' m'}(-1)^{m+m'}\begin{pmatrix}\ell & L & \ell' \\ -m & M & -m'\end{pmatrix}[f^{x,({\rm XY})}_{\ell L\ell'}]^*\overline{X}_{\ell m}\overline{Y}_{\ell' m'}\,, \tag{181}$$

with

$$f^{x,(\Theta E)}_{\ell L\ell'} = p_{\ell L\ell'}C^{\Theta{\rm B}}_{\ell}W^{x,-}_{\ell' L\ell} \tag{182}$$
$$f^{x,(\Theta B)}_{\ell L\ell'} = W^{x,0}_{\ell L\ell'}C^{\Theta{\rm B}}_{\ell'} + p_{\ell L\ell'}W^{x,+}_{\ell' L\ell}C^{\Theta{\rm B}}_{\ell}\,, \tag{183}$$
$$f^{x,(EE)}_{\ell L\ell'} = W^{x,-}_{\ell L\ell'}C^{\rm EB}_{\ell'} + p_{\ell L\ell'}W^{x,-}_{\ell' L\ell}C^{\rm EB}_{\ell}\,, \tag{184}$$
$$f^{x,(EB)}_{\ell L\ell'} = W^{x,+}_{\ell L\ell'}C^{\rm EB}_{\ell'} + p_{\ell L\ell'}W^{x,+}_{\ell' L\ell}C^{\rm EB}_{\ell}\,, \tag{185}$$
$$f^{x,(BB)}_{\ell L\ell'} = -f^{x,(EE)}_{\ell L\ell'}\,. \tag{186}$$

Note that

$$(-1)^m\begin{pmatrix}\ell & L & \ell' \\ -m & M & m'\end{pmatrix}W^{x,0}_{\ell L\ell'} = \int {\rm d}^2\hat{\boldsymbol{n}}\; Y^*_{\ell m}(\boldsymbol{\nabla}Y_{LM})\odot_x \boldsymbol{\nabla}Y_{\ell' m'}\,, \tag{187}$$
$$(-1)^m\begin{pmatrix}\ell & L & \ell' \\ -m & M & m'\end{pmatrix}W^{x,\pm 2}_{\ell L\ell'} = \int {\rm d}^2\hat{\boldsymbol{n}}\; (Y^{\pm 2}_{\ell m})^*(\boldsymbol{\nabla}Y_{LM})\odot_x \boldsymbol{\nabla}Y^{\pm 2}_{\ell' m'}\,, \tag{188}$$

and

$$(-1)^{m+m'}\begin{pmatrix}\ell & L & \ell' \\ -m & M & -m'\end{pmatrix}W^{x,+}_{\ell L\ell'} = \frac{1}{2}\int {\rm d}^2\hat{\boldsymbol{n}}\; (\boldsymbol{\nabla}Y_{LM})\odot_x\left[(Y^{+2}_{\ell m}\boldsymbol{\nabla}Y^{-2}_{\ell' m'})^* + (Y^{-2}_{\ell m}\boldsymbol{\nabla}Y^{+2}_{\ell' m'})^*\right]\,, \tag{189}$$
$$(-1)^{m+m'}\begin{pmatrix}\ell & L & \ell' \\ -m & M & -m'\end{pmatrix}W^{x,-}_{\ell L\ell'} = \frac{\rm i}{2}\int {\rm d}^2\hat{\boldsymbol{n}}\; (\boldsymbol{\nabla}Y_{LM})\odot_x\left[(Y^{+2}_{\ell m}\boldsymbol{\nabla}Y^{-2}_{\ell' m'})^* - (Y^{-2}_{\ell m}\boldsymbol{\nabla}Y^{+2}_{\ell' m'})^*\right]\,. \tag{190}$$

4.4.1 Spin fields

$$\begin{aligned}
\Theta^+_1 + {\rm i}\Theta^-_1 &\equiv -\sum_{\ell m} Y^1_{\ell m}\overline{\Theta}_{\ell m}C^{\Theta{\rm B}}_{\ell}\sqrt{(\ell+2)(\ell-1)}\,,\\
\Theta^+_3 + {\rm i}\Theta^-_3 &\equiv -\sum_{\ell m} Y^3_{\ell m}\overline{\Theta}_{\ell m}C^{\Theta{\rm B}}_{\ell}\sqrt{(\ell-2)(\ell+3)}\,,\\
\mathcal{Q}^B + {\rm i}\mathcal{U}^B &\equiv -\sum_{\ell m}\sqrt{\ell(\ell+1)}Y^1_{\ell m}C^{\Theta{\rm B}}_{\ell}{\rm i}\overline{B}_{\ell m}\,,\\
\mathcal{E}^+_1 + {\rm i}\mathcal{E}^-_1 &\equiv -\sum_{\ell m} Y^1_{\ell m}\overline{E}_{\ell m}C^{\rm EB}_{\ell}\sqrt{(\ell+2)(\ell-1)}\,,\\
\mathcal{E}^+_3 + {\rm i}\mathcal{E}^-_3 &\equiv -\sum_{\ell m} Y^3_{\ell m}\overline{E}_{\ell m}C^{\rm EB}_{\ell}\sqrt{(\ell-2)(\ell+3)}\,,\\
\mathcal{B}^+_1 + {\rm i}\mathcal{B}^-_1 &\equiv -\sum_{\ell m} Y^1_{\ell m}{\rm i}\overline{B}_{\ell m}C^{\rm EB}_{\ell}\sqrt{(\ell+2)(\ell-1)}\,,\\
\mathcal{B}^+_3 + {\rm i}\mathcal{B}^-_3 &\equiv -\sum_{\ell m} Y^3_{\ell m}{\rm i}\overline{B}_{\ell m}C^{\rm EB}_{\ell}\sqrt{(\ell-2)(\ell+3)}
\end{aligned} \tag{191}$$

4.4.2 ΘE

$$\begin{aligned}
\overline{x}_{\ell m}^{(\Theta E)} &= \sum_{\ell m}\sum_{\ell' m'}(-1)^{m+m'}\begin{pmatrix}\ell & L & \ell' \\ -m & M & -m'\end{pmatrix} p_{\ell L \ell'} C_\ell^{\Theta \mathrm{B}}[W^{x,-}_{\ell' L \ell}]^* \overline{\Theta}_{\ell m}\overline{E}_{\ell' m'} \\
&= \sum_{\ell m}\sum_{\ell' m'}(-1)^{m+m'}\begin{pmatrix}\ell' & L & \ell \\ -m' & M & -m\end{pmatrix} C_\ell^{\Theta \mathrm{B}}[W^{x,-}_{\ell' L \ell}]^* \overline{\Theta}_{\ell m}\overline{E}_{\ell' m'} \\
&= \frac{1}{2\mathrm{i}}\sum_{\ell m}\sum_{\ell' m'}\int \mathrm{d}^2\hat{\boldsymbol{n}}\, (\boldsymbol{\nabla} Y_{LM})^* \odot_x (Y^{+2}_{\ell' m'}\boldsymbol{\nabla} Y^{-2}_{\ell m} - Y^{-2}_{\ell' m'}\boldsymbol{\nabla} Y^{+2}_{\ell m}) C_\ell^{\Theta \mathrm{B}}\overline{\Theta}_{\ell m}\overline{E}_{\ell' m'} \\
&= \int \mathrm{d}^2\hat{\boldsymbol{n}}\, (\boldsymbol{\nabla} Y_{LM})^* \odot_x \frac{1}{2\mathrm{i}}(\overline{P}^E \boldsymbol{\nabla}\overline{\Theta}_B^{-2} - (\overline{P}^E)^*\boldsymbol{\nabla}\overline{\Theta}_B^{+2}) \\
&= \int \mathrm{d}^2\hat{\boldsymbol{n}}\, (\boldsymbol{\nabla} Y_{LM})^* \odot_x \Im(\overline{P}^E \boldsymbol{\nabla}\overline{\Theta}_B^{-2})\,. \qquad (192)
\end{aligned}$$

Thus, we obtain

$$\begin{aligned}
\widetilde{\boldsymbol{v}}^{\Theta E} &= \Im(\overline{P}^E \boldsymbol{\nabla}\overline{\Theta}_B^{-2}) \\
&= \Im\left[(\overline{Q}^E + \mathrm{i}\overline{U}^E)\sum_{\ell m} C_\ell^{\Theta \mathrm{B}}\overline{\Theta}_{\ell m}\left(-\sqrt{\frac{(\ell+2)(\ell-1)}{2}}Y^{-1}_{\ell m}\boldsymbol{e}^* + \sqrt{\frac{(\ell-2)(\ell+3)}{2}}Y^{-3}_{\ell m}\boldsymbol{e}\right)\right] \\
&= \frac{1}{\sqrt{2}}\Im\left[(\overline{Q}^E + \mathrm{i}\overline{U}^E)[-(\Theta_1^+ - \mathrm{i}\Theta_1^-)\boldsymbol{e}^* + (\Theta_3^+ - \mathrm{i}\Theta_3^-)\boldsymbol{e}]\right] \\
&= \frac{1}{2\sqrt{2}\mathrm{i}}\left[(\overline{Q}^E + \mathrm{i}\overline{U}^E)[-(\Theta_1^+ - \mathrm{i}\Theta_1^-)\boldsymbol{e}^* + (\Theta_3^+ - \mathrm{i}\Theta_3^-)\boldsymbol{e}] - (\overline{Q}^E - \mathrm{i}\overline{U}^E)[-(\Theta_1^+ + \mathrm{i}\Theta_1^-)\boldsymbol{e} + (\Theta_3^+ + \mathrm{i}\Theta_3^-)\boldsymbol{e}^*]\right]\,. \qquad (193)
\end{aligned}$$

We find

$$\begin{aligned}
\widetilde{v}^{\Theta E} = \sqrt{2}\boldsymbol{e}\cdot\widetilde{\boldsymbol{v}}^{\Theta E} &= \frac{1}{2\mathrm{i}}[(\overline{Q}^E + \mathrm{i}\overline{U}^E)(-\Theta_1^+ + \mathrm{i}\Theta_1^-) - (\overline{Q}^E - \mathrm{i}\overline{U}^E)(\Theta_3^+ + \mathrm{i}\Theta_3^-)] \\
&= \frac{1}{2\mathrm{i}}[\overline{Q}^E(-\Theta_1^+ - \Theta_3^+) + \overline{U}^E(-\Theta_1^- - \Theta_3^-) + \mathrm{i}\overline{Q}^E(\Theta_1^- - \Theta_3^-) + \mathrm{i}\overline{U}^E(-\Theta_1^- + \Theta_3^-)] \\
&= \frac{1}{2}[-\overline{Q}^E(\Theta_1^- - \Theta_3^-) - \overline{U}^E(-\Theta_1^- + \Theta_3^-) + \mathrm{i}\overline{Q}^E(-\Theta_1^+ - \Theta_3^+) + \mathrm{i}\overline{U}^E(-\Theta_1^- - \Theta_3^-)]\,. \qquad (194)
\end{aligned}$$

4.4.3 ΘB

$$\overline{x}_{\ell m}^{(\Theta B)} = \sum_{\ell m}\sum_{\ell' m'}(-1)^{m+m'}\begin{pmatrix}\ell & L & \ell' \\ -m & M & -m'\end{pmatrix}[W^{x,0}_{\ell L \ell'}C_{\ell'}^{\Theta \mathrm{B}} + p_{\ell L \ell'}W^{x,+}_{\ell' L \ell}C_\ell^{\Theta \mathrm{B}}]^*\overline{\Theta}_{\ell m}\overline{B}_{\ell' m'}\,. \qquad (195)$$

The first term becomes

$$\begin{aligned}
\overline{x}_{\ell m}^{(\Theta B)}|_{1\mathrm{st}} &= \sum_{\ell m}\sum_{\ell' m'}(-1)^{m+m'}\begin{pmatrix}\ell & L & \ell' \\ -m & M & -m'\end{pmatrix}[W^{x,0}_{\ell L \ell'}C_{\ell'}^{\Theta \mathrm{B}}]^*\overline{\Theta}_{\ell m}\overline{B}_{\ell' m'} \\
&= \sum_{\ell m}\sum_{\ell' m'}\int \mathrm{d}^2\hat{\boldsymbol{n}}\, (\boldsymbol{\nabla} Y_{LM})^* \odot_x (Y_{\ell m}\boldsymbol{\nabla} Y_{\ell' m'}) C_{\ell'}^{\Theta \mathrm{B}}\overline{\Theta}_{\ell m}\overline{B}_{\ell' m'} \\
&= \int \mathrm{d}^2\hat{\boldsymbol{n}}\, (\boldsymbol{\nabla} Y_{LM})^* \odot_x \overline{\Theta}(-\mathrm{i})\sum_{\ell' m'}\sqrt{\frac{\ell'(\ell'+1)}{2}}\left(-Y^1_{\ell' m'}\boldsymbol{e}^* + Y^{-1}_{\ell' m'}\boldsymbol{e}\right) C_{\ell'}^{\Theta \mathrm{B}}\mathrm{i}\overline{B}_{\ell' m'} \\
&= \int \mathrm{d}^2\hat{\boldsymbol{n}}\, (\boldsymbol{\nabla} Y_{LM})^* \odot_x \overline{\Theta}\frac{-\mathrm{i}}{\sqrt{2}}[(\mathcal{Q}^B + \mathrm{i}\mathcal{U}^B)\boldsymbol{e}^* - (\mathcal{Q}^B - \mathrm{i}\mathcal{U}^B)\boldsymbol{e}]\,. \qquad (196)
\end{aligned}$$

The second term becomes

$$\begin{aligned}
\overline{x}^{(\Theta B)}_{\ell m}|_{\rm 2nd} &= \sum_{\ell m}\sum_{\ell' m'}(-1)^{m+m'}\begin{pmatrix}\ell & L & \ell' \\ -m & M & -m'\end{pmatrix} p_{\ell L\ell'}(W^{x,+}_{\ell' L\ell})^* C^{\Theta\rm B}_\ell \overline{\Theta}_{\ell m}\overline{B}_{\ell' m'} \\
&= \sum_{\ell m}\sum_{\ell' m'}(-1)^{m+m'}\begin{pmatrix}\ell' & L & \ell \\ -m' & M & -m\end{pmatrix} (W^{x,+}_{\ell' L\ell})^* C^{\Theta\rm B}_\ell \overline{\Theta}_{\ell m}\overline{B}_{\ell' m'} \\
&= \sum_{\ell m}\sum_{\ell' m'}\frac{1}{2}\int {\rm d}^2\hat{\boldsymbol{n}}\ (\boldsymbol{\nabla} Y_{LM})^*\odot_x\left[Y^{+2}_{\ell' m'}\boldsymbol{\nabla} Y^{-2}_{\ell m} + Y^{-2}_{\ell' m'}\boldsymbol{\nabla} Y^{+2}_{\ell m}\right] C^{\Theta\rm B}_\ell \overline{\Theta}_{\ell m}\overline{B}_{\ell' m'} \\
&= \int {\rm d}^2\hat{\boldsymbol{n}}\ (\boldsymbol{\nabla} Y_{LM})^*\odot_x \frac{1}{2{\rm i}}\left[\overline{P}^B\sum_{\ell m}\boldsymbol{\nabla} Y^{-2}_{\ell m} C^{\Theta\rm B}_\ell\overline{\Theta}_{\ell m} - {\rm c.c.}\right] \\
&= \widetilde{v}^{\Theta E}|_{\overline{P}^E\to\overline{P}^B}\,. \qquad (197)
\end{aligned}$$

Thus, we obtain

$$\widetilde{v}^{\Theta B} = -{\rm i}\overline{\Theta}(\mathcal{Q}^B+{\rm i}\mathcal{U}^B) + \frac{1}{2}[-\overline{Q}^B(\Theta_1^- - \Theta_3^-) - \overline{U}^B(-\Theta_1^- + \Theta_3^-) + {\rm i}\overline{Q}^B(-\Theta_1^+ - \Theta_3^+) + {\rm i}\overline{U}^B(-\Theta_1^- - \Theta_3^-)]\,. \qquad (198)$$

4.4.4 EE

$$\begin{aligned}
\overline{x}^{(EE)}_{\ell m} &= \frac{1}{2}\sum_{\ell m}\sum_{\ell' m'}(-1)^{m+m'}\begin{pmatrix}\ell & L & \ell' \\ -m & M & -m'\end{pmatrix}[W^{x,-}_{\ell L\ell'}C^{\rm EB}_{\ell'} + p_{\ell L\ell'}W^{x,-}_{\ell' L\ell}C^{\rm EB}_\ell]^*\overline{E}_{\ell m}\overline{E}_{\ell' m'} \\
&= \frac{1}{2}\sum_{\ell m}\sum_{\ell' m'}\frac{\rm i}{2}\int {\rm d}^2\hat{\boldsymbol{n}}\ (\boldsymbol{\nabla} Y_{LM})^*\odot_x\left[Y^{+2}_{\ell m}\boldsymbol{\nabla} Y^{-2}_{\ell' m'} - Y^{-2}_{\ell m}\boldsymbol{\nabla} Y^{+2}_{\ell' m'} + (\ell\leftrightarrow\ell')\right] C^{\rm EB}_{\ell'}\overline{E}_{\ell m}\overline{E}_{\ell' m'} \\
&= \sum_{\ell m}\sum_{\ell' m'}\frac{\rm i}{2}\int {\rm d}^2\hat{\boldsymbol{n}}\ (\boldsymbol{\nabla} Y_{LM})^*\odot_x\left[Y^{+2}_{\ell m}\boldsymbol{\nabla} Y^{-2}_{\ell' m'} - Y^{-2}_{\ell m}\boldsymbol{\nabla} Y^{+2}_{\ell' m'}\right] C^{\rm EB}_{\ell'}\overline{E}_{\ell m}\overline{E}_{\ell' m'} \\
&= \int {\rm d}^2\hat{\boldsymbol{n}}\ (\boldsymbol{\nabla} Y_{LM})^*\odot_x\frac{\rm i}{2}\left[\overline{P}^E\sum_{\ell m}C^{\rm EB}_\ell\overline{E}_{\ell m}\Big(-\sqrt{\frac{(\ell+2)(\ell-1)}{2}}Y^{-1}_{\ell m}\boldsymbol{e}^* + \sqrt{\frac{(\ell-2)(\ell+3)}{2}}Y^{-3}_{\ell m}\boldsymbol{e}\Big) - {\rm c.c.}\right] \\
&= \int {\rm d}^2\hat{\boldsymbol{n}}\ (\boldsymbol{\nabla} Y_{LM})^*\odot_x\frac{\rm i}{2\sqrt{2}}\left[\overline{P}^E((-\mathcal{E}_1^+ + {\rm i}\mathcal{E}_1^-)\boldsymbol{e}^* + (\mathcal{E}_3^+ - {\rm i}\mathcal{E}_3^-)\boldsymbol{e}) - {\rm c.c.}\right]\,. \qquad (199)
\end{aligned}$$

Thus, we find

$$\begin{aligned}
\widetilde{v}^{EE} &= \frac{\rm i}{2}[(\overline{Q}^E + {\rm i}\overline{U}^E)(-\mathcal{E}_1^+ + {\rm i}\mathcal{E}_1^-) - (\overline{Q}^E - {\rm i}\overline{U}^E)(\mathcal{E}_3^+ + {\rm i}\mathcal{E}_3^-)] \\
&= \frac{\rm i}{2}[-\overline{Q}^E(\mathcal{E}_1^+ + \mathcal{E}_3^+) - \overline{U}^E(\mathcal{E}_1^- + \mathcal{E}_3^-) + {\rm i}\overline{Q}^E(\mathcal{E}_1^- - \mathcal{E}_3^-) + {\rm i}\overline{U}^E(-\mathcal{E}_1^+ + \mathcal{E}_3^+)] \\
&= \frac{-1}{2}[\overline{Q}^E(\mathcal{E}_1^- - \mathcal{E}_3^-) + \overline{U}^E(-\mathcal{E}_1^+ + \mathcal{E}_3^+) + {\rm i}\overline{Q}^E(\mathcal{E}_1^+ + \mathcal{E}_3^+) + {\rm i}\overline{U}^E(\mathcal{E}_1^- + \mathcal{E}_3^-)]\,. \qquad (200)
\end{aligned}$$

4.4.5 EB

$$\begin{aligned}
\overline{x}^{(EB)}_{\ell m} &= \sum_{\ell m}\sum_{\ell' m'}(-1)^{m+m'}\begin{pmatrix}\ell & L & \ell' \\ -m & M & -m'\end{pmatrix}[W^{x,+}_{\ell L\ell'}C^{\rm EB}_{\ell'} + p_{\ell L\ell'}W^{x,+}_{\ell' L\ell}C^{\rm EB}_\ell]^*\overline{E}_{\ell m}\overline{B}_{\ell' m'} \\
&= \sum_{\ell m}\sum_{\ell' m'}\frac{1}{2}\int {\rm d}^2\hat{\boldsymbol{n}}\ (\boldsymbol{\nabla} Y_{LM})^*\odot_x\left[Y^{+2}_{\ell m}\boldsymbol{\nabla} Y^{-2}_{\ell' m'} + Y^{-2}_{\ell m}\boldsymbol{\nabla} Y^{+2}_{\ell' m'}\right] C^{\rm EB}_{\ell'}\overline{E}_{\ell m}\overline{B}_{\ell' m'} + (E\leftrightarrow B) \\
&= \int {\rm d}^2\hat{\boldsymbol{n}}\ (\boldsymbol{\nabla} Y_{LM})^*\odot_x(-1)\Im\left[\overline{P}^E\sum_{\ell' m'}C^{\rm EB}_{\ell'}{\rm i}\overline{B}_{\ell' m'}\boldsymbol{\nabla} Y^{-2}_{\ell' m'} + \overline{P}^B\sum_{\ell' m'}C^{\rm EB}_{\ell'}\overline{E}_{\ell' m'}\boldsymbol{\nabla} Y^{-2}_{\ell' m'}\right] \\
&= \int {\rm d}^2\hat{\boldsymbol{n}}\ (\boldsymbol{\nabla} Y_{LM})^*\odot_x(-1)\Im\left[\overline{P}^E((\mathcal{B}_1^+ - {\rm i}\mathcal{B}_1^-)\boldsymbol{e}^* - (\mathcal{B}_3^+ - {\rm i}\mathcal{B}_3^-)\boldsymbol{e})\right] - (E\leftrightarrow {\rm i}B)\,. \qquad (201)
\end{aligned}$$

Thus, we find

$$\widetilde{v}^{EB} = \frac{\mathrm{i}}{2}\bigg[(\overline{Q}^E + \mathrm{i}\overline{U}^E)(\mathcal{B}_1^+ - \mathrm{i}\mathcal{B}_1^-) + (\overline{Q}^E - \mathrm{i}\overline{U}^E)(\mathcal{B}_3^+ + \mathrm{i}\mathcal{B}_3^-) - (\overline{Q}^B + \mathrm{i}\overline{U}^B)(\mathcal{E}_1^+ - \mathrm{i}\mathcal{E}_1^-) - (\overline{Q}^B - \mathrm{i}\overline{U}^B)(\mathcal{E}_3^+ + \mathrm{i}\mathcal{E}_3^-)\bigg] \tag{202}$$

$$= \frac{\mathrm{i}}{2}\bigg[\overline{Q}^E(\mathcal{B}_1^+ + \mathcal{B}_3^+) + \overline{U}^E(\mathcal{B}_1^- + \mathcal{B}_3^-) + \overline{Q}^B(\mathcal{E}_1^+ + \mathcal{E}_3^+) + \overline{U}^B(\mathcal{E}_1^- + \mathcal{E}_3^-) + \mathrm{i}\overline{Q}^E(-\mathcal{B}_1^- + \mathcal{B}_3^-) + \mathrm{i}\overline{U}^E(\mathcal{B}_1^+ - \mathcal{B}_3^+) - \mathrm{i}\overline{Q}^B(-\mathcal{E}_1^- + \mathcal{E}_3^-) - \mathrm{i}\overline{U}^B(\mathcal{E}_1^+ - \mathcal{E}_3^+)\bigg] . \tag{203}$$

4.4.6 BB

By replacing $\overline{E}_{\ell m}$ to $\overline{B}_{\ell m}$, we find:

$$\widetilde{v}^{BB} = \frac{1}{2}[\overline{Q}^B(\mathcal{B}_1^- - \mathcal{B}_3^-) + \overline{U}^B(-\mathcal{B}_1^+ + \mathcal{B}_3^+) + \mathrm{i}\overline{Q}^B(\mathcal{B}_1^+ + \mathcal{B}_3^+) + \mathrm{i}\overline{U}^B(\mathcal{B}_1^- + \mathcal{B}_3^-)] . \tag{204}$$

4.5 Polarization angle and amplitude modulation

The polarization rotation and amplitude estimators are related each other since the former and later estimate the imaginary and real parts of the multiplicative fields, respectively. Explicitly, we can obtain the amplitude estimator by changing the operation of taking the imaginary/real part with that of taking the real/imaginary part and then by multiplying $1/2$. Here, we explicitly derive the estimator for the amplitude modulation.

4.5.1 $\Theta\Theta$

The unnormalized estimator for $\Theta\Theta$ is given by

$$[\overline{\epsilon}^{\Theta\Theta}_{LM}]^* = \frac{1}{2}\sum_{\ell\ell' mm'}\begin{pmatrix}\ell & \ell' & L\\ m & m' & M\end{pmatrix} W^{\epsilon,0}_{\ell L\ell'}(C^{\Theta\Theta}_{\ell'} + C^{\Theta\Theta}_{\ell})\overline{\Theta}_{\ell m}\overline{\Theta}_{\ell' m'} , \tag{205}$$

and the sum is non-zero only when $\ell + L + \ell'$ is even. The estimator contains

$$\begin{pmatrix}\ell & \ell' & L\\ m & m' & M\end{pmatrix} W^{\epsilon,0}_{\ell L\ell'} = \int \mathrm{d}\hat{\boldsymbol{n}}\; Y_{\ell m}Y_{LM}Y_{\ell' m'} , \tag{206}$$

Substituting the above equation to Eq. (205), we obtain the unnormalzied estimator as

$$\overline{\epsilon}^{\Theta\Theta}_{LM} = \sum_{\ell\ell' mm'}\int \mathrm{d}\hat{\boldsymbol{n}}\; Y^*_{\ell m}Y^*_{LM}Y^*_{\ell' m'}\frac{[C^{\Theta\Theta}_{\ell'} + C^{\Theta\Theta}_{\ell}]}{2}\overline{\Theta}^*_{\ell m}\overline{\Theta}^*_{\ell' m'}$$

$$= \int \mathrm{d}\hat{\boldsymbol{n}}\; Y^*_{LM}\left[\sum_{\ell m}\overline{\Theta}_{\ell m}Y_{\ell m}\right]\left[\sum_{\ell' m'}C^{\Theta\Theta}_{\ell'}\overline{\Theta}_{\ell' m'}Y_{\ell' m'}\right] . \tag{207}$$

4.5.2 ΘE

The ΘE quadratic unnormalized estimator for the amplitude modulation is given by

$$[\overline{\epsilon}^{\Theta \mathrm{E}}_{LM}]^* = \sum_{\ell\ell' mm'}\begin{pmatrix}\ell & \ell' & L\\ m & m' & M\end{pmatrix}\left[W^{\epsilon,0}_{\ell L\ell'}C^{\Theta \mathrm{E}}_{\ell'} + W^{\epsilon,+}_{\ell' L\ell}C^{\Theta \mathrm{E}}_{\ell}\right]\overline{\Theta}_{\ell m}\overline{E}_{\ell' m'} . \tag{208}$$

Using the property of the Wigner 3j,

$$\begin{aligned}\begin{pmatrix}\ell & \ell' & L\\ m & m' & M\end{pmatrix} W^{\epsilon,+}_{\ell' L\ell} &= \frac{[1+(-1)^{\ell+L+\ell'}]}{2}\gamma_{\ell L\ell'}\begin{pmatrix}\ell' & L & \ell\\ 2 & 0 & -2\end{pmatrix}\begin{pmatrix}\ell & \ell' & L\\ m & m' & M\end{pmatrix}\\ &= \frac{1}{2}\gamma_{\ell L\ell'}\left[\begin{pmatrix}\ell & \ell' & L\\ -2 & 2 & 0\end{pmatrix}+\begin{pmatrix}\ell & \ell' & L\\ 2 & -2 & 0\end{pmatrix}\right]\begin{pmatrix}\ell & \ell' & L\\ m & m' & M\end{pmatrix}\\ &= \frac{1}{2}\int \mathrm{d}\hat{\boldsymbol{n}}\ Y_{LM}[Y^2_{\ell m}Y^{-2}_{\ell' m'}+Y^{-2}_{\ell m}Y^2_{\ell' m'}]\,. \end{aligned} \tag{209}$$

Using the above equation and Eq. (206), we obtain

$$\begin{aligned}\overline{\epsilon}^{\Theta\mathrm{E}}_{LM} &= \frac{1}{2}\int \mathrm{d}\hat{\boldsymbol{n}}\ Y_{LM}\sum_{\ell\ell' mm'}\{2Y_{\ell m}Y_{\ell' m'}C^{\Theta\mathrm{E}}_{\ell'}+(Y^2_{\ell m}Y^{-2}_{\ell' m'}+Y^{-2}_{\ell m}Y^2_{\ell' m'})C^{\Theta\mathrm{E}}_{\ell}\}\overline{\Theta}_{\ell m}\overline{E}_{\ell' m'}\\ &= \frac{1}{2}\int \mathrm{d}\hat{\boldsymbol{n}}\ Y_{LM}\left\{\sum_{\ell m}Y_{\ell m}\overline{\Theta}_{\ell m}\sum_{\ell' m'}Y_{\ell' m'}C^{\Theta\mathrm{E}}_{\ell'}\overline{E}_{\ell' m'}+\sum_{\ell m}Y^2_{\ell m}C^{\Theta\mathrm{E}}_{\ell}\overline{\Theta}_{\ell m}\sum_{\ell' m'}Y^{-2}_{\ell' m'}\overline{E}_{\ell' m'}+\text{c.c.}\right\}\\ &= \int \mathrm{d}\hat{\boldsymbol{n}}\ Y_{LM}\Re[\Theta^0(E^{0,+}+\mathrm{i}E^{0,-})+(\Theta^{2,+}+\mathrm{i}\Theta^{2,-})(Q^E-\mathrm{i}U^E)]\\ &= \int \mathrm{d}\hat{\boldsymbol{n}}\ Y_{LM}[\Theta^0E^{0,+}+\Theta^{2,+}Q^E+\Theta^{2,-}U^E]\,, \end{aligned} \tag{210}$$

where

$$\Theta^{2,+}+\mathrm{i}\Theta^{2,-} = -\sum_{\ell m}Y^2_{\ell m}C^{\Theta\mathrm{E}}_{\ell}\overline{\Theta}_{\ell m}\,, \tag{211}$$

$$E^{0,+}+\mathrm{i}E^{0,-} = \sum_{\ell' m'}Y_{\ell' m'}C^{\Theta\mathrm{E}}_{\ell'}\overline{E}_{\ell' m'}\,. \tag{212}$$

4.5.3 ΘB

The ΘB quadratic unnormalized estimator for the amplitude modulation is given by

$$[\overline{\epsilon}^{\Theta\mathrm{B}}_{LM}]^* = \sum_{\ell\ell' mm'}\begin{pmatrix}\ell & \ell' & L\\ m & m' & M\end{pmatrix} W^{\epsilon,-}_{\ell' L\ell}C^{\Theta\mathrm{E}}_{\ell}\overline{\Theta}_{\ell m}\overline{B}_{\ell' m'}\,. \tag{213}$$

Using the property of the Wigner 3j, we obtain

$$\begin{aligned}\begin{pmatrix}\ell & \ell' & L\\ m & m' & M\end{pmatrix} W^{\epsilon,-}_{\ell' L\ell} &= \mathrm{i}\frac{[1-(-1)^{\ell+L+\ell'}]}{2}\gamma_{\ell L\ell'}\begin{pmatrix}\ell' & L & \ell\\ 2 & 0 & -2\end{pmatrix}\begin{pmatrix}\ell & \ell' & L\\ m & m' & M\end{pmatrix}\\ &= \frac{\mathrm{i}}{2}\gamma_{\ell L\ell'}\left[\begin{pmatrix}\ell & \ell' & L\\ -2 & 2 & 0\end{pmatrix}-\begin{pmatrix}\ell & \ell' & L\\ 2 & -2 & 0\end{pmatrix}\right]\begin{pmatrix}\ell & \ell' & L\\ m & m' & M\end{pmatrix}\\ &= \frac{\mathrm{i}}{2}\int \mathrm{d}\hat{\boldsymbol{n}}\ Y_{LM}[Y^2_{\ell m}Y^{-2}_{\ell' m'}-Y^{-2}_{\ell m}Y^2_{\ell' m'}]\,. \end{aligned} \tag{214}$$

We then obtain

$$\begin{aligned}\overline{\epsilon}^{\Theta\mathrm{B}}_{LM} &= \frac{\mathrm{i}}{2}\int \mathrm{d}\hat{\boldsymbol{n}}\ Y_{LM}\sum_{\ell\ell' mm'}(Y^2_{\ell m}Y^{-2}_{\ell' m'}-Y^{-2}_{\ell m}Y^2_{\ell' m'})C^{\Theta\mathrm{E}}_{\ell}\overline{\Theta}_{\ell m}\overline{B}_{\ell' m'}\\ &= \int \mathrm{d}\hat{\boldsymbol{n}}\ Y_{LM}\Re\left(\sum_{\ell m}Y^2_{\ell m}C^{\Theta\mathrm{E}}_{\ell}\overline{\Theta}_{\ell m}\sum_{\ell' m'}Y^{-2}_{\ell' m'}\mathrm{i}\overline{B}_{\ell' m'}\right)\,. \end{aligned} \tag{215}$$

4.5.4 EE

The EE quadratic unnormalized estimator for the amplitude modulation is given by

$$[\overline{\epsilon}^{\mathrm{EE}}_{LM}]^* = \frac{1}{2}\sum_{\ell\ell' mm'}\begin{pmatrix}\ell & \ell' & L\\ m & m' & M\end{pmatrix} W^{\epsilon,+}_{\ell L\ell'}(C^{\mathrm{EE}}_{\ell'}+C^{\mathrm{EE}}_{\ell})\overline{E}_{\ell m}\overline{E}_{\ell' m'}\,. \tag{216}$$

Using Eq. (209), we obtain

$$\begin{aligned}\overline{\epsilon}^{\rm EE}_{LM} &= \frac{1}{4}\int {\rm d}\hat{\boldsymbol{n}}\ Y_{LM} \sum_{\ell\ell' mm'} [Y^2_{\ell m}Y^{-2}_{\ell' m'} + Y^{-2}_{\ell m}Y^2_{\ell' m'}]\left[\overline{E}_{\ell m}(C^{\rm EE}_{\ell'}\overline{E}_{\ell' m'}) + (C^{\rm EE}_{\ell}\overline{E}_{\ell m})\overline{E}_{\ell' m'}\right] \\ &= \frac{1}{4}\int {\rm d}\hat{\boldsymbol{n}}\ Y^*_{LM}\Big[\sum_{\ell m} Y^2_{\ell m}\overline{E}_{\ell m}\sum_{\ell' m'} Y^{-2}_{\ell' m'}C^{\rm EE}_{\ell'}\overline{E}_{\ell' m'} + \sum_{\ell m} Y^{-2}_{\ell m}\overline{E}_{\ell m}\sum_{\ell' m'} Y^{2}_{\ell' m'}C^{\rm EE}_{\ell'}\overline{E}_{\ell' m'} \\ &\quad + \sum_{\ell m} Y^2_{\ell m}C^{\rm EE}_{\ell}\overline{E}_{\ell m}\sum_{\ell' m'} Y^{-2}_{\ell' m'}\overline{E}_{\ell' m'} + \sum_{\ell m} Y^{-2}_{\ell m}C^{\rm EE}_{\ell}\overline{E}_{\ell m}\sum_{\ell' m'} Y^{2}_{\ell' m'}\overline{E}_{\ell' m'}\Big] \\ &= \frac{1}{2}\int {\rm d}\hat{\boldsymbol{n}}\ Y^*_{LM}\Big[\sum_{\ell m} Y^2_{\ell m}\overline{E}_{\ell m}\sum_{\ell' m'} Y^{-2}_{\ell' m'}C^{\rm EE}_{\ell'}\overline{E}_{\ell' m'} + \sum_{\ell m} Y^{-2}_{\ell m}\overline{E}_{\ell m}\sum_{\ell' m'} Y^{2}_{\ell' m'}C^{\rm EE}_{\ell'}\overline{E}_{\ell' m'}\Big]\,. \end{aligned} \tag{217}$$

4.5.5 EB

The EB quadratic unnormalized estimator for the amplitude modulation is given by

$$[\overline{\epsilon}^{\rm EB}_{LM}]^* = \sum_{\ell\ell' mm'} \begin{pmatrix} \ell & \ell' & L \\ m & m' & M \end{pmatrix} \left[W^{\epsilon,-}_{\ell L\ell'}C^{\rm BB}_{\ell'} + W^{\epsilon,-}_{\ell' L\ell}C^{\rm EE}_{\ell}\right]\overline{E}_{\ell m}\overline{B}_{\ell' m'}\,. \tag{218}$$

Note that $W^{\epsilon,-}_{\ell L\ell'} = W^{\epsilon,-}_{\ell' L\ell}$. Using Eq. (214), we obtain

$$\begin{aligned}\overline{\epsilon}^{\rm EB}_{LM} &= \frac{\rm i}{2}\int {\rm d}\hat{\boldsymbol{n}}\ Y^*_{LM}\Big[\sum_{\ell m} Y^2_{\ell m}\overline{E}_{\ell m}\sum_{\ell' m'} Y^{-2}_{\ell' m'}C^{\rm BB}_{\ell'}\overline{B}_{\ell' m'} - \sum_{\ell m} Y^{-2}_{\ell m}\overline{E}_{\ell m}\sum_{\ell' m'} Y^{2}_{\ell' m'}C^{\rm BB}_{\ell'}\overline{B}_{\ell' m'} \\ &\quad + \sum_{\ell m} Y^{-2}_{\ell m}C^{\rm EE}_{\ell}\overline{E}_{\ell m}\sum_{\ell' m'} Y^{2}_{\ell' m'}\overline{B}_{\ell' m'} - \sum_{\ell m} Y^{2}_{\ell m}C^{\rm EE}_{\ell}\overline{E}_{\ell m}\sum_{\ell' m'} Y^{-2}_{\ell' m'}\overline{B}_{\ell' m'}\Big] \\ &= \int {\rm d}\hat{\boldsymbol{n}}\ Y^*_{LM}\Re\left[\sum_{\ell m} Y^2_{\ell m}\overline{E}_{\ell m}\sum_{\ell' m'} Y^{-2}_{\ell' m'}C^{\rm BB}_{\ell'}{\rm i}\overline{B}_{\ell' m'} + \sum_{\ell m} Y^{2}_{\ell m}C^{\rm EE}_{\ell}\overline{E}_{\ell m}\sum_{\ell' m'} Y^{-2}_{\ell' m'}{\rm i}\overline{B}_{\ell' m'}\right]. \end{aligned} \tag{219}$$

4.5.6 EB (rotation)

The EB quadratic estimator for the polarization rotation is given by

$$[\overline{\alpha}^{\rm EB}_{LM}]^* = \sum_{\ell\ell' mm'} \begin{pmatrix} \ell & \ell' & L \\ m & m' & M \end{pmatrix} \left[W^{\alpha,-}_{\ell L\ell'}C^{\rm BB}_{\ell'} - W^{\alpha,-}_{\ell' L\ell}C^{\rm EE}_{\ell}\right]\overline{E}_{\ell m}\overline{B}_{\ell' m'}\,. \tag{220}$$

Using Eqs. (68) and (209), we obtain

$$\begin{aligned}\overline{\alpha}^{\rm EB}_{LM} &= -\int {\rm d}\hat{\boldsymbol{n}}\ Y^*_{LM}\Big[\sum_{\ell m} Y^{-2}_{\ell m}\overline{E}_{\ell m}\sum_{\ell' m'} Y^{2}_{\ell' m'}C^{\rm BB}_{\ell'}\overline{B}_{\ell' m'} + \sum_{\ell m} Y^{2}_{\ell m}\overline{E}_{\ell m}\sum_{\ell' m'} Y^{-2}_{\ell' m'}C^{\rm BB}_{\ell'}\overline{B}_{\ell' m'} \\ &\quad - \sum_{\ell m} Y^{2}_{\ell m}C^{\rm EE}_{\ell}\overline{E}_{\ell m}\sum_{\ell' m'} Y^{-2}_{\ell' m'}\overline{B}_{\ell' m'} - \sum_{\ell m} Y^{-2}_{\ell m}C^{\rm EE}_{\ell}\overline{E}_{\ell m}\sum_{\ell' m'} Y^{2}_{\ell' m'}\overline{B}_{\ell' m'}\Big] \\ &= -\int {\rm d}\hat{\boldsymbol{n}}\ Y^*_{LM}\Big[\sum_{\ell m} Y^{-2}_{\ell m}\overline{E}_{\ell m}\sum_{\ell' m'} Y^{2}_{\ell' m'}C^{\rm BB}_{\ell'}\overline{B}_{\ell' m'} - \sum_{\ell m} Y^{-2}_{\ell m}C^{\rm EE}_{\ell}\overline{E}_{\ell m}\sum_{\ell' m'} Y^{2}_{\ell' m'}\overline{B}_{\ell' m'} + {\rm c.c.}\Big] \\ &= {\rm i}\int {\rm d}\hat{\boldsymbol{n}}\ Y^*_{LM}\left[(Q^E - {\rm i}U^E)(\mathcal{Q}^B + {\rm i}\mathcal{U}^B) - (\mathcal{Q}^E - {\rm i}\mathcal{U}^E)(Q^B + {\rm i}U^B) - {\rm c.c.}\right] \\ &= -2\int {\rm d}\hat{\boldsymbol{n}}\ Y^*_{LM}\Im\left[(Q^E - {\rm i}U^E)(\mathcal{Q}^B + {\rm i}\mathcal{U}^B) - (\mathcal{Q}^E - {\rm i}\mathcal{U}^E)(Q^B + {\rm i}U^B)\right] \\ &= -2\int {\rm d}\hat{\boldsymbol{n}}\ Y^*_{LM}\left[Q^E\mathcal{U}^B - U^E\mathcal{Q}^B - \mathcal{Q}^E U^B + \mathcal{U}^E Q^B\right]\,. \end{aligned} \tag{221}$$

where we define

$$\mathcal{Q}^E + {\rm i}\mathcal{U}^E = -\sum_{\ell m} Y^2_{\ell m}C^{\rm EE}_{\ell}\overline{E}_{\ell m}\,, \tag{222}$$

$$\mathcal{Q}^B + {\rm i}\mathcal{U}^B = -\sum_{\ell m} Y^2_{\ell m}C^{\rm BB}_{\ell}{\rm i}\overline{B}_{\ell m}\,. \tag{223}$$

4.5.7 EB (odd)

The EB quadratic estimator for the amplitude modulation is given by

$$[\overline{\epsilon}^{\mathrm{EB},-}_{LM}]^* = 2\sum_{\ell\ell'mm'}\begin{pmatrix}\ell & \ell' & L\\ m & m' & M\end{pmatrix}\left[C^{\mathrm{EB}}_{\ell'}W^{\epsilon,+}_{\ell L\ell'} + C^{\mathrm{EB}}_{\ell}W^{\epsilon,+}_{\ell' L\ell}\right]\overline{E}_{\ell m}\overline{B}_{\ell' m'}\,. \tag{224}$$

Note that $W^{\alpha,-}_{\ell' L\ell} = -2W^{\epsilon,+}_{\ell' L\ell}$. We then obtain the estimator by replacing $C^{\mathrm{BB}}_{\ell'}$ and C^{EE}_{ℓ} with $-C^{\mathrm{EB}}_{\ell'}$ and C^{EB}_{ℓ}, respectively, in the polarization rotation estimator, and multiplying $1/2$, yielding

$$\overline{\epsilon}^{\mathrm{EB},-}_{LM} = -2\int\mathrm{d}\hat{\boldsymbol{n}}\; Y^*_{LM}\left[-Q^E\mathcal{U}^B + U^E\mathcal{Q}^B + Q^B\mathcal{U}^E - U^B\mathcal{Q}^E\right]\,. \tag{225}$$

where we define

$$\begin{aligned}\mathcal{Q}^B + \mathrm{i}\mathcal{U}^B &= -\sum_{\ell m} Y^2_{\ell m}C^{\mathrm{EB}}_{\ell}\mathrm{i}\overline{B}_{\ell m}\\ \mathcal{Q}^E + \mathrm{i}\mathcal{U}^E &= -\sum_{\ell m} Y^2_{\ell m}C^{\mathrm{EB}}_{\ell}\overline{E}_{\ell m}\,.\end{aligned} \tag{226}$$

5 Quadratic estimator normalization, noise spectrum and diagonal RDN0

Here, we generalize the algorithm of [15] to the case including the cosmic bi-refringence, patchy reionization, and so on. We also discuss implementation of the diagonal RDN0 [16] using the normalization code.

I provide expression of the noise covariance between unnormalized estimators. We define true observed spectra as, $\widehat{\mathcal{C}}^{\Theta\Theta}$, $\widehat{\mathcal{C}}^{\Theta E}$, $\widehat{\mathcal{C}}^{EE}$ and $\widehat{\mathcal{C}}^{BB}$. For forecasts, you can assume $\widehat{\mathcal{C}}=\widehat{C}$. On the other hand, for the application of the diagonal RDN0 calculation, we should distinguish between the true spectrum, $\widehat{\mathcal{C}}$, and our assumption on the observed spectrum, $\widehat{C}$.

With $s=0,\pm$, we first define the following kernel functions;

$$\Sigma^{(s),xy}_L[A,B] = \frac{1}{2L+1}\sum_{\ell\ell'}(W^{x,s}_{\ell L\ell'})^* W^{y,s}_{\ell L\ell'}A_\ell B_{\ell'}\,, \tag{227}$$

$$\Sigma^{(\times),xy}_L[A,B] = \frac{1}{2L+1}\sum_{\ell\ell'}W^{x,0}_{\ell L\ell'} W^{y,+}_{\ell L\ell'}A_\ell B_{\ell'}\,, \tag{228}$$

$$\Gamma^{(s),xy}_L[A,B] = \frac{1}{2L+1}\sum_{\ell\ell'}(W^{x,s}_{\ell L\ell'})^* W^{y,s}_{\ell' L\ell}A_\ell B_{\ell'}\,, \tag{229}$$

$$\Gamma^{(\times),xy}_L[A,B] = \frac{1}{2L+1}\sum_{\ell\ell'}W^{x,0}_{\ell L\ell'} W^{y,+}_{\ell' L\ell}A_\ell B_{\ell'}\,. \tag{230}$$

The above functions are real. Note that

$$\Sigma^{(s),xy}_L[A,B] = \Sigma^{(s),yx}_L[A,B]\,, \tag{231}$$

$$\Gamma^{(s),xy}_L[A,B] = \Gamma^{(s),yx}_L[B,A]\,. \tag{232}$$

Note also that $p_x=p_y$ otherwise the noise spectrum vanishes due to the parity symmetry. We denote $p_x=p_y=p$.

5.1 Disconnected noise spectrum, normalization and response function

5.1.1 ΘΘ

The noise spectrum between $\widehat{x}^{\Theta\Theta}$ and $\widehat{y}^{\Theta\Theta}$ is given by

$$\begin{aligned} N^{xy,(\Theta\Theta)}_L &= \frac{1}{2L+1}\sum_{\ell\ell'}\frac{\left[W^{x,0}_{\ell L\ell'}C^{\Theta\Theta}_{\ell'}+pW^{x,0}_{\ell' L\ell}C^{\Theta\Theta}_{\ell}\right]\left[W^{y,0}_{\ell L\ell'}C^{\Theta\Theta}_{\ell'}+pW^{y,0}_{\ell' L\ell}C^{\Theta\Theta}_{\ell}\right]}{2\widehat{C}^{\Theta\Theta}_\ell\widehat{C}^{\Theta\Theta}_{\ell'}}\frac{\widehat{\mathcal{C}}^{\Theta\Theta}_\ell}{\widehat{C}^{\Theta\Theta}_\ell}\frac{\widehat{\mathcal{C}}^{\Theta\Theta}_{\ell'}}{\widehat{C}^{\Theta\Theta}_{\ell'}} \\ &= \Sigma^{(0),xy}_L\left[\frac{1}{\widehat{C}^{\Theta\Theta}},\frac{(C^{\Theta\Theta})^2}{\widehat{C}^{\Theta\Theta}}\right]+p\Gamma^{(0),xy}_L\left[\frac{C^{\Theta\Theta}}{\widehat{C}^{\Theta\Theta}},\frac{C^{\Theta\Theta}}{\widehat{C}^{\Theta\Theta}}\right]\,. \end{aligned} \tag{233}$$

Here, we assume $\widehat{\mathcal{C}}=\widehat{C}$ from the first to the second line. For example, if either x or y are point sources,

$$\begin{aligned} N^{xs,(\Theta\Theta)}_L &= \frac{1}{2L+1}\sum_{\ell\ell'}\frac{\left[W^{x,0}_{\ell L\ell'}C^{\Theta\Theta}_{\ell'}+W^{x,0}_{\ell' L\ell}C^{\Theta\Theta}_{\ell}\right]\left[W^{\epsilon,0}_{\ell L\ell'}+W^{\epsilon,0}_{\ell' L\ell}\right]}{4\widehat{C}^{\Theta\Theta}_\ell\widehat{C}^{\Theta\Theta}_{\ell'}} \\ &= \frac{1}{2}\Sigma^{(0),x\epsilon}_L\left[\frac{1}{\widehat{C}^{\Theta\Theta}},\frac{C^{\Theta\Theta}}{\widehat{C}^{\Theta\Theta}}\right]+\frac{1}{2}\Gamma^{(0),x\epsilon}_L\left[\frac{1}{\widehat{C}^{\Theta\Theta}},\frac{C^{\Theta\Theta}}{\widehat{C}^{\Theta\Theta}}\right]\,. \end{aligned} \tag{234}$$

In the idealistic case, the normalization and noise spectrum are identical. Substituting $x=y$ into Eq. (233) gives the normalization of $\widehat{x}$:

$$[A^{x,(\Theta\Theta)}_L]^{-1} = \Sigma^{(0),xx}_L\left[\frac{1}{\widehat{C}^{\Theta\Theta}},\frac{(C^{\Theta\Theta})^2}{\widehat{C}^{\Theta\Theta}}\right]+p\Gamma^{(0),xx}_L\left[\frac{C^{\Theta\Theta}}{\widehat{C}^{\Theta\Theta}},\frac{C^{\Theta\Theta}}{\widehat{C}^{\Theta\Theta}}\right]\,. \tag{235}$$

The response to y is given by

$$R^{x^{(\Theta\Theta)},y}_L = \Sigma^{(0),xy}_L\left[\frac{1}{\widehat{C}^{\Theta\Theta}},\frac{(C^{\Theta\Theta})^2}{\widehat{C}^{\Theta\Theta}}\right]+p\Gamma^{(0),xy}_L\left[\frac{C^{\Theta\Theta}}{\widehat{C}^{\Theta\Theta}},\frac{C^{\Theta\Theta}}{\widehat{C}^{\Theta\Theta}}\right]\,. \tag{236}$$

5.1.2 ΘE

The normalization of the quadratic ΘE estimator is given by

$$[A_L^{x,(\Theta E)}]^{-1} = \frac{1}{2L+1}\sum_{\ell\ell'}\frac{|W^{x,0}_{\ell L\ell'}C^{\Theta\mathrm{E}}_{\ell'}+pW^{x,+}_{\ell' L\ell}C^{\Theta\mathrm{E}}_{\ell}|^2}{\widehat{C}^{\Theta\Theta}_\ell\widehat{C}^{\mathrm{EE}}_{\ell'}}$$
$$= \frac{1}{2L+1}\sum_{\ell\ell'}\left[(W^{x,0}_{\ell L\ell'})^2\frac{(C^{\Theta\mathrm{E}}_{\ell'})^2}{\widehat{C}^{\Theta\Theta}_\ell\widehat{C}^{\mathrm{EE}}_{\ell'}}+2pW^{x,0}_{\ell L\ell'}W^{x,+}_{\ell' L\ell}\frac{C^{\Theta\mathrm{E}}_{\ell'}C^{\Theta\mathrm{E}}_{\ell}}{\widehat{C}^{\Theta\Theta}_\ell\widehat{C}^{\mathrm{EE}}_{\ell'}}+(W^{x,+}_{\ell' L\ell})^2\frac{(C^{\Theta\mathrm{E}}_{\ell})^2}{\widehat{C}^{\Theta\Theta}_\ell\widehat{C}^{\mathrm{EE}}_{\ell'}}\right]$$
$$= \Sigma^{(0),x}_L\left[\frac{1}{\widehat{C}^{\Theta\Theta}},\frac{(C^{\Theta\mathrm{E}})^2}{\widehat{C}^{\mathrm{EE}}}\right]+2\Gamma^{(\times),x}_L\left[\frac{C^{\Theta\mathrm{E}}}{\widehat{C}^{\Theta\Theta}},\frac{C^{\Theta\mathrm{E}}}{\widehat{C}^{\mathrm{EE}}}\right]+\Sigma^{(+),x}_L\left[\frac{1}{\widehat{C}^{\mathrm{EE}}},\frac{(C^{\Theta\mathrm{E}})^2}{\widehat{C}^{\Theta\Theta}}\right], \quad (237)$$

and for the imaginary counterpart:

$$[A_L^{x,(\Theta E)}]^{-1} = \frac{1}{2L+1}\sum_{\ell\ell'}\frac{|W^{x,-}_{\ell' L\ell}C^{\Theta\mathrm{B}}_{\ell}|^2}{\widehat{C}^{\Theta\Theta}_\ell\widehat{C}^{\mathrm{EE}}_{\ell'}} = \Sigma^{(+),x}_L\left[\frac{1}{\widehat{C}^{\mathrm{EE}}},\frac{(C^{\Theta\mathrm{B}})^2}{\widehat{C}^{\Theta\Theta}}\right]. \quad (238)$$

5.1.3 ΘB

The normalization of the quadratic ΘB estimator is given by

$$[A_L^{x,(\Theta B)}]^{-1} = \frac{1}{2L+1}\sum_{\ell\ell'}\frac{|W^{x,-}_{\ell' L\ell}C^{\Theta\mathrm{E}}_{\ell}|^2}{\widehat{C}^{\Theta\Theta}_\ell\widehat{C}^{\mathrm{BB}}_{\ell'}} = \Sigma^{(-),x}_L\left[\frac{1}{\widehat{C}^{\mathrm{BB}}},\frac{(C^{\Theta\mathrm{E}})^2}{\widehat{C}^{\Theta\Theta}}\right], \quad (239)$$

and for the imaginary counterpart:

$$[A_L^{x,(\Theta B)}]^{-1} = \frac{1}{2L+1}\sum_{\ell\ell'}\frac{|W^{x,0}_{\ell L\ell'}C^{\Theta\mathrm{B}}_{\ell'}+pW^{x,+}_{\ell' L\ell}C^{\Theta\mathrm{B}}_{\ell}|^2}{\widehat{C}^{\Theta\Theta}_\ell\widehat{C}^{\mathrm{BB}}_{\ell'}}$$
$$= \Sigma^{(0),x}_L\left[\frac{1}{\widehat{C}^{\Theta\Theta}},\frac{(C^{\Theta\mathrm{B}})^2}{\widehat{C}^{\mathrm{BB}}}\right]+2p\Gamma^{(\times),x}_L\left[\frac{C^{\Theta\mathrm{E}}}{\widehat{C}^{\Theta\Theta}},\frac{C^{\Theta\mathrm{B}}}{\widehat{C}^{\mathrm{BB}}}\right]+\Sigma^{(+),x}_L\left[\frac{1}{\widehat{C}^{\mathrm{BB}}},\frac{(C^{\Theta\mathrm{B}})^2}{\widehat{C}^{\Theta\Theta}}\right]. \quad (240)$$

5.1.4 EE and BB

The normalization of the quadratic EE estimator (and for the BB estimator by replacing the $EE\to BB$ spectrum) is given by

$$[A_L^{x,(EE)}]^{-1} = \frac{1}{2L+1}\sum_{\ell\ell'}\frac{|W^{x,+}_{\ell L\ell'}C^{\mathrm{EE}}_{\ell'}+pW^{x,+}_{\ell' L\ell}C^{\mathrm{EE}}_{\ell}|^2}{2\widehat{C}^{\mathrm{EE}}_\ell\widehat{C}^{\mathrm{EE}}_{\ell'}}$$
$$= \Sigma^{(+),x}_L\left[\frac{1}{\widehat{C}^{\mathrm{EE}}},\frac{(C^{\mathrm{EE}})^2}{\widehat{C}^{\mathrm{EE}}}\right]+p\Gamma^{(+),x}_L\left[\frac{C^{\mathrm{EE}}}{\widehat{C}^{\mathrm{EE}}},\frac{C^{\mathrm{EE}}}{\widehat{C}^{\mathrm{EE}}}\right]. \quad (241)$$

The imaginary counterpart is given by

$$[A_L^{\tilde{x},(EE)}]^{-1} = \frac{1}{2L+1}\sum_{\ell\ell'}\frac{|W^{x,-}_{\ell L\ell'}C^{\mathrm{EB}}_{\ell'}-pW^{x,-}_{\ell' L\ell}C^{\mathrm{EB}}_{\ell}|^2}{2\widehat{C}^{\mathrm{EE}}_\ell\widehat{C}^{\mathrm{EE}}_{\ell'}}$$
$$= \Sigma^{(-),x}_L\left[\frac{1}{\widehat{C}^{\mathrm{EE}}},\frac{(C^{\mathrm{EB}})^2}{\widehat{C}^{\mathrm{EE}}}\right]-p\Gamma^{(-),x}_L\left[\frac{C^{\mathrm{EB}}}{\widehat{C}^{\mathrm{EE}}},\frac{C^{\mathrm{EB}}}{\widehat{C}^{\mathrm{EE}}}\right], \quad (242)$$

$$[A_L^{\tilde{x},(BB)}]^{-1} = \frac{1}{2L+1}\sum_{\ell\ell'}\frac{|W^{x,-}_{\ell L\ell'}C^{\mathrm{EB}}_{\ell'}-pW^{x,-}_{\ell' L\ell}C^{\mathrm{EB}}_{\ell}|^2}{2\widehat{C}^{\mathrm{BB}}_\ell\widehat{C}^{\mathrm{BB}}_{\ell'}}$$
$$= \Sigma^{(-),x}_L\left[\frac{1}{\widehat{C}^{\mathrm{BB}}},\frac{(C^{\mathrm{EB}})^2}{\widehat{C}^{\mathrm{BB}}}\right]-p\Gamma^{(-),x}_L\left[\frac{C^{\mathrm{EB}}}{\widehat{C}^{\mathrm{BB}}},\frac{C^{\mathrm{EB}}}{\widehat{C}^{\mathrm{BB}}}\right], \quad (243)$$

5.1.5 EB

The noise spectrum between the (even) quadratic EB estimators is given by

$$
\begin{aligned}
N_L^{xy,(EB)} &= \frac{1}{2L+1}\sum_{\ell\ell'}\frac{(W^{x,-}_{\ell L\ell'}C^{\rm BB}_{\ell'}+pW^{x,-}_{\ell' L\ell}C^{\rm EE}_{\ell})(W^{y,-}_{\ell L\ell'}C^{\rm BB}_{\ell'}+pW^{y,-}_{\ell' L\ell}C^{\rm EE}_{\ell})}{\widehat{C}^{\rm EE}_\ell\widehat{C}^{\rm BB}_{\ell'}} \\
&= \Sigma^{(-),xy}_L\left[\frac{1}{\widehat{C}^{\rm EE}},\frac{(C^{\rm BB})^2}{\widehat{C}^{\rm BB}}\right] \\
&\quad + 2p\Gamma^{(-),xy}_L\left[\frac{C^{\rm BB}}{\widehat{C}^{\rm BB}},\frac{C^{\rm EE}}{\widehat{C}^{\rm EE}}\right] + \Sigma^{(-),xy}_L\left[\frac{1}{\widehat{C}^{\rm BB}},\frac{(C^{\rm EE})^2}{\widehat{C}^{\rm EE}}\right] \,,
\end{aligned}
\tag{244}
$$

Substituting $x=y$ into the above equation gives the normalization of the quadratic EB estimator:

$$
[A^{x,(EB)}_L]^{-1} = \Sigma^{(-),x}_L\left[\frac{1}{\widehat{C}^{\rm EE}},\frac{(C^{\rm BB})^2}{\widehat{C}^{\rm BB}}\right] + 2p\Gamma^{(-),x}_L\left[\frac{C^{\rm EE}}{\widehat{C}^{\rm EE}},\frac{C^{\rm BB}}{\widehat{C}^{\rm BB}}\right] + \Sigma^{(-),x}_L\left[\frac{1}{\widehat{C}^{\rm BB}},\frac{(C^{\rm EE})^2}{\widehat{C}^{\rm EE}}\right] \,,
\tag{245}
$$

The imaginary counterpart is given by

$$
[A^{x,(EB),-}_L]^{-1} = \Sigma^{(+),x}_L\left[\frac{1}{\widehat{C}^{\rm EE}},\frac{(C^{\rm EB})^2}{\widehat{C}^{\rm BB}}\right] + 2p\Gamma^{(+),x}_L\left[\frac{C^{\rm EB}}{\widehat{C}^{\rm EE}},\frac{C^{\rm EB}}{\widehat{C}^{\rm BB}}\right] + \Sigma^{(+),x}_L\left[\frac{1}{\widehat{C}^{\rm BB}},\frac{(C^{\rm EB})^2}{\widehat{C}^{\rm EE}}\right] \,.
\tag{246}
$$

5.2 Off-diagonal noise spectrum

5.2.1 $\Theta\Theta\Theta E$

The noise spectrum between $\widehat{x}^{\Theta\Theta}$ and $\widehat{y}^{\Theta E}$ is given by

$$
\begin{aligned}
N_L^{x^{\Theta\Theta}y^{\Theta E}} &= \frac{1}{2L+1}\sum_{\ell\ell'}\left[\frac{W^{x,0}_{\ell L\ell'}C^{\Theta\Theta}_{\ell'}}{2\widehat{C}^{\Theta\Theta}_\ell\widehat{C}^{\Theta\Theta}_{\ell'}}+p_x(\ell\leftrightarrow\ell')\right]\left[\frac{(W^{y,0}_{\ell L\ell'}C^{\Theta\rm E}_{\ell'}+p_yW^{y,+}_{\ell' L\ell}C^{\Theta\rm E}_{\ell})}{\widehat{C}^{\Theta\Theta}_\ell\widehat{C}^{\rm EE}_{\ell'}}\widehat{\mathcal{C}}^{\Theta\Theta}_\ell\widehat{\mathcal{C}}^{\Theta E}_{\ell'}+p_y(\ell\leftrightarrow\ell')\right] \\
&= \frac{1}{2L+1}\sum_{\ell\ell'}\left[\frac{1+p_xp_y}{2}\frac{W^{x,0}_{\ell L\ell'}C^{\Theta\Theta}_{\ell'}}{\widehat{C}^{\Theta\Theta}_\ell\widehat{C}^{\Theta\Theta}_{\ell'}}\frac{(W^{y,0}_{\ell L\ell'}C^{\Theta\rm E}_{\ell'}+p_yW^{y,+}_{\ell' L\ell}C^{\Theta\rm E}_{\ell})}{\widehat{C}^{\Theta\Theta}_\ell\widehat{C}^{\rm EE}_{\ell'}}\widehat{\mathcal{C}}^{\Theta\Theta}_\ell\widehat{\mathcal{C}}^{\Theta E}_{\ell'}\right. \\
&\quad \left. +\frac{p_x+p_y}{2}\frac{W^{x,0}_{\ell L\ell'}C^{\Theta\Theta}_{\ell'}}{\widehat{C}^{\Theta\Theta}_\ell\widehat{C}^{\Theta\Theta}_{\ell'}}\frac{(W^{y,0}_{\ell' L\ell}C^{\Theta\rm E}_{\ell}+p_yW^{y,+}_{\ell L\ell'}C^{\Theta\rm E}_{\ell'})}{\widehat{C}^{\Theta\Theta}_{\ell'}\widehat{C}^{\rm EE}_{\ell}}\widehat{\mathcal{C}}^{\Theta\Theta}_{\ell'}\widehat{\mathcal{C}}^{\Theta E}_{\ell}\right] \\
&= \Sigma^{(0),xy}_L\left[\frac{\widehat{\mathcal{C}}^{\Theta\Theta}}{(\widehat{C}^{\Theta\Theta})^2},\frac{C^{\Theta\Theta}C^{\Theta\rm E}\widehat{\mathcal{C}}^{\Theta E}}{\widehat{C}^{\Theta\Theta}\widehat{C}^{\rm EE}}\right] + p\Gamma^{(\times),xy}_L\left[\frac{C^{\Theta\rm E}\widehat{\mathcal{C}}^{\Theta\Theta}}{(\widehat{C}^{\Theta\Theta})^2},\frac{C^{\Theta\Theta}\widehat{C}^{\Theta\rm E}}{\widehat{C}^{\Theta\Theta}\widehat{C}^{\rm EE}}\right] \\
&\quad + p\Gamma^{(0),xy}_L\left[\frac{C^{\Theta\rm E}\widehat{\mathcal{C}}^{\Theta E}}{\widehat{C}^{\Theta\Theta}\widehat{C}^{\rm EE}},\frac{C^{\Theta\Theta}\widehat{\mathcal{C}}^{\Theta\Theta}}{(\widehat{C}^{\Theta\Theta})^2}\right] + \Sigma^{(\times),xy}_L\left[\frac{\widehat{\mathcal{C}}^{\Theta E}}{\widehat{C}^{\Theta\Theta}\widehat{C}^{\rm EE}},\frac{C^{\Theta\rm E}C^{\Theta\Theta}\widehat{\mathcal{C}}^{\Theta\Theta}}{(\widehat{C}^{\Theta\Theta})^2}\right] \,,
\end{aligned}
\tag{247}
$$

Note that the code assumes $\widehat{\mathcal{C}}=\widehat{C}$. For diagonal RDN0, we should change the input as follows:

$$
\widehat{C}^{\Theta\Theta} \to (\widehat{C}^{\Theta\Theta}/\widehat{\mathcal{C}}^{\Theta\Theta})\widehat{C}^{\Theta\Theta} \,,
\tag{248}
$$

$$
\widehat{C}^{\rm EE} \to (\widehat{C}^{\Theta\Theta}/\widehat{\mathcal{C}}^{\Theta\Theta})\widehat{C}^{\rm EE} \,.
\tag{249}
$$

5.2.2 $\Theta\Theta EE$

The noise spectrum between the $\Theta\Theta$ and EE estimators is given by

$$\begin{aligned}
N_L^{x^{\Theta\Theta}y^{EE}} &= \frac{1}{2L+1}\sum_{\ell\ell'}\left[\frac{W^{x,0}_{\ell L\ell'}C^{\Theta\Theta}_{\ell'}}{2\widehat{C}^{\Theta\Theta}_{\ell}\widehat{C}^{\Theta\Theta}_{\ell'}}+p_x(\ell\leftrightarrow\ell')\right]\left[\frac{(W^{y,+}_{\ell L\ell'}C^{\rm EE}_{\ell'}+p_yW^{y,+}_{\ell' L\ell}C^{\rm EE}_{\ell})}{2\widehat{C}^{\rm EE}_{\ell}\widehat{C}^{\rm EE}_{\ell'}}\widehat{\mathcal{C}}^{\Theta E}_{\ell}\widehat{\mathcal{C}}^{\Theta E}_{\ell'}+p_y(\ell\leftrightarrow\ell')\right] \\
&= \frac{1}{2L+1}\sum_{\ell\ell'}\frac{1+p_xp_y}{2}\frac{W^{x,0}_{\ell L\ell'}C^{\Theta\Theta}_{\ell'}}{\widehat{C}^{\Theta\Theta}_{\ell}\widehat{C}^{\Theta\Theta}_{\ell'}}\left[\frac{(W^{y,+}_{\ell L\ell'}C^{\rm EE}_{\ell'}+p_yW^{y,+}_{\ell' L\ell}C^{\rm EE}_{\ell})}{2\widehat{C}^{\rm EE}_{\ell}\widehat{C}^{\rm EE}_{\ell'}}\widehat{\mathcal{C}}^{\Theta E}_{\ell}\widehat{\mathcal{C}}^{\Theta E}_{\ell'}+p_y(\ell\leftrightarrow\ell')\right] \\
&= \frac{1}{2L+1}\sum_{\ell\ell'}\frac{1+p_xp_y}{2}\frac{W^{x,0}_{\ell L\ell'}C^{\Theta\Theta}_{\ell'}}{\widehat{C}^{\Theta\Theta}_{\ell}\widehat{C}^{\Theta\Theta}_{\ell'}}\frac{(W^{y,+}_{\ell L\ell'}C^{\rm EE}_{\ell'}+p_yW^{y,+}_{\ell' L\ell}C^{\rm EE}_{\ell})}{\widehat{C}^{\rm EE}_{\ell}\widehat{C}^{\rm EE}_{\ell'}}\widehat{\mathcal{C}}^{\Theta E}_{\ell}\widehat{\mathcal{C}}^{\Theta E}_{\ell'} \\
&= \Sigma^{(\times),xy}_L\left[\frac{\widehat{\mathcal{C}}^{\Theta E}}{\widehat{C}^{\Theta\Theta}\widehat{C}^{\rm EE}},\frac{C^{\Theta\Theta}C^{\rm EE}\widehat{\mathcal{C}}^{\Theta E}}{\widehat{C}^{\Theta\Theta}\widehat{C}^{\rm EE}}\right]+p\Gamma^{(\times),xy}_L\left[\frac{\widehat{\mathcal{C}}^{\Theta E}C^{\rm EE}}{\widehat{C}^{\Theta\Theta}\widehat{C}^{\rm EE}},\frac{C^{\Theta\Theta}\widehat{\mathcal{C}}^{\Theta E}}{\widehat{C}^{\Theta\Theta}\widehat{C}^{\rm EE}}\right]\,.
\end{aligned}\tag{250}$$

5.2.3 ΘEEE

The noise spectrum between the ΘE and EE estimators is given by

$$\begin{aligned}
N_L^{x^{\Theta E}y^{EE}} &= \frac{1}{2L+1}\sum_{\ell\ell'}\left[\frac{(W^{x,0}_{\ell L\ell'}C^{\Theta\rm E}_{\ell'}+p_xW^{x,+}_{\ell' L\ell}C^{\Theta\rm E}_{\ell})}{\widehat{C}^{\Theta\Theta}_{\ell}\widehat{C}^{\rm EE}_{\ell'}}\widehat{\mathcal{C}}^{\Theta E}_{\ell}\widehat{\mathcal{C}}^{EE}_{\ell'}+p_x(\ell\leftrightarrow\ell')\right]\left[\frac{W^{y,+}_{\ell L\ell'}C^{\rm EE}_{\ell'}}{2\widehat{C}^{\rm EE}_{\ell}\widehat{C}^{\rm EE}_{\ell'}}+p_y(\ell\leftrightarrow\ell')\right] \\
&= \frac{1}{2L+1}\sum_{\ell\ell'}\left[\frac{(W^{x,0}_{\ell L\ell'}C^{\Theta\rm E}_{\ell'}+p_xW^{x,+}_{\ell' L\ell}C^{\Theta\rm E}_{\ell})}{\widehat{C}^{\Theta\Theta}_{\ell}\widehat{C}^{\rm EE}_{\ell'}}\widehat{\mathcal{C}}^{\Theta E}_{\ell}\widehat{\mathcal{C}}^{EE}_{\ell'}\right]\left[\frac{W^{y,+}_{\ell L\ell'}C^{\rm EE}_{\ell'}}{\widehat{C}^{\rm EE}_{\ell}\widehat{C}^{\rm EE}_{\ell'}}+p_y\frac{W^{y,+}_{\ell' L\ell}C^{\rm EE}_{\ell}}{\widehat{C}^{\rm EE}_{\ell}\widehat{C}^{\rm EE}_{\ell'}}\right] \\
&= \Sigma^{(\times),xy}_L\left[\frac{\widehat{\mathcal{C}}^{\Theta E}}{\widehat{C}^{\Theta\Theta}\widehat{C}^{\rm EE}},\frac{C^{\Theta\rm E}C^{\rm EE}\widehat{\mathcal{C}}^{EE}}{(\widehat{C}^{\rm EE})^2}\right]+p\Gamma^{(+),yx}_L\left[\frac{C^{\Theta\rm E}\widehat{\mathcal{C}}^{\Theta E}}{\widehat{C}^{\Theta\Theta}\widehat{C}^{\rm EE}},\frac{C^{\rm EE}\widehat{\mathcal{C}}^{EE}}{(\widehat{C}^{\rm EE})^2}\right] \\
&\quad +p\Gamma^{(\times),xy}_L\left[\frac{\widehat{\mathcal{C}}^{\Theta E}C^{\rm EE}}{\widehat{C}^{\Theta\Theta}\widehat{C}^{\rm EE}},\frac{C^{\Theta\rm E}\widehat{\mathcal{C}}^{EE}}{(\widehat{C}^{\rm EE})^2}\right]+\Sigma^{(+),xy}_L\left[\frac{\widehat{\mathcal{C}}^{EE}}{(\widehat{C}^{\rm EE})^2},\frac{C^{\Theta\rm E}C^{\rm EE}\widehat{\mathcal{C}}^{\Theta E}}{\widehat{C}^{\Theta\Theta}\widehat{C}^{\rm EE}}\right]\,.
\end{aligned}\tag{251}$$

For the diagonal RDN0, we should change the input as follows:

$$\widehat{C}^{\rm EE}\to(\widehat{C}^{\rm EE}/\widehat{\mathcal{C}}^{EE})\widehat{C}^{\rm EE}\,,\tag{252}$$

$$\widehat{C}^{\Theta\Theta}\to(\widehat{\mathcal{C}}^{EE}/\widehat{C}^{\rm EE})\widehat{C}^{\Theta\Theta}\,.\tag{253}$$

5.2.4 ΘBEB

The noise spectrum between the ΘB and EB estimators is given by

$$\begin{aligned}
N_L^{x^{\Theta B}y^{EB}} &= \frac{1}{2L+1}\sum_{\ell\ell'}\left[\frac{p_xW^{x,-}_{\ell' L\ell}C^{\Theta\rm E}_{\ell}}{\widehat{C}^{\Theta\Theta}_{\ell}\widehat{C}^{\rm BB}_{\ell'}}\widehat{\mathcal{C}}^{\Theta E}_{\ell}\widehat{\mathcal{C}}^{BB}_{\ell'}\right]\left[\frac{(W^{y,-}_{\ell L\ell'})^*C^{\rm BB}_{\ell'}-p_y(W^{y,-}_{\ell' L\ell})^*C^{\rm EE}_{\ell}}{\widehat{C}^{\rm EE}_{\ell}\widehat{C}^{\rm BB}_{\ell'}}\right] \\
&= p\Gamma^{(-),yx}_L\left[\frac{C^{\Theta\rm E}\widehat{\mathcal{C}}^{\Theta E}}{\widehat{C}^{\Theta\Theta}\widehat{C}^{\rm EE}},\frac{C^{\rm BB}\widehat{\mathcal{C}}^{BB}}{(\widehat{C}^{\rm BB})^2}\right]+\Sigma^{(-),xy}_L\left[\frac{\widehat{\mathcal{C}}^{BB}}{(\widehat{C}^{\rm BB})^2},\frac{C^{\Theta\rm E}C^{\rm EE}\widehat{\mathcal{C}}^{\Theta E}}{\widehat{C}^{\Theta\Theta}\widehat{C}^{\rm EE}}\right]\,.
\end{aligned}\tag{254}$$

For diagonal RDN0, we should change the input as follows:

$$\widehat{C}^{\rm BB}\to(\widehat{C}^{\rm BB}/\widehat{\mathcal{C}}^{BB})\widehat{C}^{\rm BB}\,,\tag{255}$$

5.3 Noise covariance and diagonal RDN0

5.3.1 General expression

Let us consider the disconnected four point correlation involved in the (cross-)power spectrum between $\overline{x}^{XY}_{LM}$ and $\overline{x}^{ZW}_{LM}$ derived in [12]:

$$\langle \overline{x}^{(XY)}_{LM} (\overline{x}^{(ZW)}_{LM})^* \rangle = \frac{1}{2L+1} \sum_{\ell\ell'} (g^{XY}_{\ell L \ell'})^* \left[g^{ZW}_{\ell L \ell'} \widehat{\mathcal{C}}^{XZ}_{\ell} \widehat{\mathcal{C}}^{YW}_{\ell'} + p_{\ell L \ell'} g^{ZW}_{\ell' L \ell} \widehat{\mathcal{C}}^{XW}_{\ell} \widehat{\mathcal{C}}^{YZ}_{\ell'} \right] . \quad (256)$$

A naive approach to evaluate the above disconnected bias is to use our best approximation to $\widehat{\mathcal{C}}$, i.e., replacing $\widehat{\mathcal{C}}$ to $\widehat{C}$ in the above equation. This approach is, however, not so accurate given that the dominant bias in the lensing power spectrum is the disconnected four point correlation. Alternatively, we can use the realization-dependent bias subtraction (RDN0) which is more robust to estimate the disconnected bias [17]. If we can assume that the off-diagonal covariance of the CMB multipoles is negligible, we can further simplify to the "diagonal" RDN0 [16]. The diagonal RDN0 is defined as:

$$\begin{aligned} \widehat{N}^{x,(XY,ZW),\mathrm{dRD}}_L &= \frac{1}{2L+1} \sum_{\ell\ell'} (g^{XY}_{\ell L \ell'})^* \\ &\quad \times \Big[g^{ZW}_{\ell L \ell'} (\widehat{\mathcal{C}}^{XZ}_{\ell} \widehat{C}^{YW}_{\ell'} + \widehat{C}^{XZ}_{\ell} \widehat{\mathcal{C}}^{YW}_{\ell'} - \widehat{C}^{XZ}_{\ell} \widehat{C}^{YW}_{\ell'}) \\ &\quad\quad + p_{\ell L \ell'} g^{ZW}_{\ell' L \ell} (\widehat{\mathcal{C}}^{XW}_{\ell} \widehat{C}^{YZ}_{\ell'} + \widehat{C}^{XW}_{\ell} \widehat{\mathcal{C}}^{YZ}_{\ell'} - \widehat{C}^{XW}_{\ell} \widehat{C}^{YZ}_{\ell'}) \Big] \end{aligned} \quad (257)$$

For the code implementation, we rewrite the above equation as:

$$\begin{aligned} \widehat{N}^{x,(XY,ZW),\mathrm{dRD}}_L &= \frac{1}{2L+1} \sum_{\ell\ell'} (g^{XY}_{\ell L \ell'})^* \\ &\quad \times \left[g^{ZW}_{\ell L \ell'} (\widehat{\mathcal{C}}^{XZ}_{\ell} \widehat{\mathcal{C}}^{YW}_{\ell'} - \Delta\widehat{\mathcal{C}}^{XZ}_{\ell} \Delta\widehat{\mathcal{C}}^{YW}_{\ell'}) + p_{\ell L \ell'} g^{ZW}_{\ell' L \ell} (\widehat{\mathcal{C}}^{XW}_{\ell} \widehat{\mathcal{C}}^{YZ}_{\ell'} - \Delta\widehat{\mathcal{C}}^{XW}_{\ell} \Delta\widehat{\mathcal{C}}^{YZ}_{\ell'}) \right] , \end{aligned} \quad (258)$$

where we define $\Delta\widehat{\mathcal{C}}^{AB}_{\ell} = \widehat{\mathcal{C}}^{AB}_{\ell} - \widehat{C}^{AB}_{\ell}$. For the diagonal noise covariance, $XY = ZW$, we obtain:

$$\begin{aligned} \widehat{N}^{x,(XY,XY),\mathrm{dRD}}_L &= \frac{1}{2L+1} \sum_{\ell\ell'} (g^{XY}_{\ell L \ell'})^* \\ &\quad \times \left[g^{XY}_{\ell L \ell'} (\widehat{\mathcal{C}}^{XX}_{\ell} \widehat{\mathcal{C}}^{YY}_{\ell'} - \Delta\widehat{\mathcal{C}}^{XX}_{\ell} \Delta\widehat{\mathcal{C}}^{YY}_{\ell'}) + p_{\ell L \ell'} g^{XY}_{\ell' L \ell} (\widehat{\mathcal{C}}^{XY}_{\ell} \widehat{\mathcal{C}}^{XY}_{\ell'} - \Delta\widehat{\mathcal{C}}^{XY}_{\ell} \Delta\widehat{\mathcal{C}}^{XY}_{\ell'}) \right] \\ &= \frac{1}{2L+1} \sum_{\ell\ell'} (g^{XY}_{\ell L \ell'})^* \frac{1}{\Delta^{XY}} \\ &\quad \times \left[f^{XY}_{\ell L \ell'} \frac{\widehat{\mathcal{C}}^{XX}_{\ell} \widehat{\mathcal{C}}^{YY}_{\ell'} - \Delta\widehat{\mathcal{C}}^{XX}_{\ell} \Delta\widehat{\mathcal{C}}^{YY}_{\ell'}}{\widehat{C}^{XX}_{\ell} \widehat{C}^{YY}_{\ell'}} + p_{\ell L \ell'} f^{XY}_{\ell' L \ell} \frac{\widehat{\mathcal{C}}^{XY}_{\ell} \widehat{\mathcal{C}}^{XY}_{\ell'} - \Delta\widehat{\mathcal{C}}^{XY}_{\ell} \Delta\widehat{\mathcal{C}}^{XY}_{\ell'}}{\widehat{C}^{XX}_{\ell'} \widehat{C}^{YY}_{\ell}} \right] . \end{aligned} \quad (259)$$

If $X = Y$, the above equation is further simplified as:

$$\widehat{N}^{x,(XX,XX),\mathrm{dRD}}_L = \frac{1}{2L+1} \sum_{\ell\ell'} \frac{1}{2} (f^{XX}_{\ell L \ell'})^* f^{XX}_{\ell L \ell'} \left(\frac{\widehat{\mathcal{C}}^{XX}_{\ell} \widehat{\mathcal{C}}^{XX}_{\ell'} - \Delta\widehat{\mathcal{C}}^{XX}_{\ell} \Delta\widehat{\mathcal{C}}^{XX}_{\ell'}}{(\widehat{C}^{XX}_{\ell} \widehat{C}^{XX}_{\ell'})^2} \right) . \quad (260)$$

If $XY = \Theta B$ or EB,

$$\widehat{N}^{x,(XY,XY),\mathrm{dRD}}_L = \frac{1}{2L+1} \sum_{\ell\ell'} (f^{XY}_{\ell L \ell'})^* f^{XY}_{\ell L \ell'} \left(\frac{\widehat{\mathcal{C}}^{XX}_{\ell} \widehat{\mathcal{C}}^{YY}_{\ell'} - \Delta\widehat{\mathcal{C}}^{XX}_{\ell} \Delta\widehat{\mathcal{C}}^{YY}_{\ell'}}{(\widehat{C}^{XX}_{\ell} \widehat{C}^{YY}_{\ell'})^2} \right) . \quad (261)$$

The above expressions are the same as the normalization but with $\widehat{\mathcal{C}}/\widehat{C}^2$ and $\Delta\widehat{\mathcal{C}}/\widehat{C}^2$ instead of $1/\widehat{C}$ to evaluate the first and second terms, respectively. The first order of the difference between the true and assumed CMB spectra, $\Delta\widehat{\mathcal{C}}$, does not appear.

5.4 Asymmetric quadratic estimators

Here we consider asymmetric quadratic estimators [18].

5.4.1 ΘΘ

The asymmetric estimator is defined as:

$$x^{\Theta^a\Theta^b}_{LM} = A^{x^{\Theta^a\Theta^b}}_L \sum_{\ell\ell' mm'} \begin{pmatrix} \ell & \ell' & L \\ m & m' & M \end{pmatrix} w^{\mathrm{L},\Theta^a}_\ell w^{\mathrm{G},\Theta^b\Theta^a}_{\ell'} W^{x,0}_{\ell L\ell'} \Theta^a_{\ell m}\Theta^b_{\ell' m'} \,, \tag{262}$$

where we define the lensed map and gradient map filters as [19]:

$$w^{\mathrm{L},X}_\ell \equiv \frac{1}{\widehat{C}^{XX}_\ell} \,, \tag{263}$$

$$w^{\mathrm{G},XY}_\ell \equiv \frac{C^{XY}_\ell}{\widehat{C}^{XX}_\ell} \,. \tag{264}$$

Note that

$$x^{(\Theta^a\Theta^b)}_{LM} \equiv \frac{x^{\Theta^a\Theta^b}_{LM} + x^{\Theta^b\Theta^a}_{LM}}{2} \to x^{\Theta\Theta}_{LM} \quad (a=b) \,. \tag{265}$$

The normalization of $\widehat{x}^{\Theta\Theta}$ for the asymmetric estimator is given by:

$$\begin{aligned} A^{x^{\Theta^a\Theta^b}}_L &= \frac{1}{2L+1}\sum_{\ell\ell'} w^{\mathrm{L},\Theta^a}_\ell w^{\mathrm{G},\Theta^b\Theta^a}_{\ell'} W^{x,0}_{\ell L\ell'} \left[W^{x,0}_{\ell L\ell'} C^{\Theta\Theta}_{\ell'} + pW^{x,0}_{\ell' L\ell} C^{\Theta\Theta}_\ell\right] \\ &= \Sigma^{(0),xx}_L \left[w^{\mathrm{L},\Theta^a}, w^{\mathrm{G},\Theta^b\Theta^a} C^{\Theta\Theta}\right] + p\Gamma^{(0),xx}_L \left[w^{\mathrm{L},\Theta^a} C^{\Theta\Theta}, w^{\mathrm{G},\Theta^b\Theta^a}\right] \,. \end{aligned} \tag{266}$$

The response to y is given by

$$\begin{aligned} R^{x^{\Theta^a\Theta^b},y}_L &= \frac{1}{2L+1}\sum_{\ell\ell'} w^{\mathrm{L},\Theta^a}_\ell w^{\mathrm{G},\Theta^b\Theta^a}_{\ell'} W^{x,0}_{\ell L\ell'} \left[W^{y,0}_{\ell L\ell'} C^{\Theta\Theta}_{\ell'} + pW^{y,0}_{\ell' L\ell} C^{\Theta\Theta}_\ell\right] \\ &= \Sigma^{(0),xy}_L \left[w^{\mathrm{L},\Theta^a}, w^{\mathrm{G},\Theta^b\Theta^a} C^{\Theta\Theta}\right] + p\Gamma^{(0),xy}_L \left[w^{\mathrm{L},\Theta^a} C^{\Theta\Theta}, w^{\mathrm{G},\Theta^b\Theta^a}\right] \,. \end{aligned} \tag{267}$$

For example,

$$\begin{aligned} R^{\phi^{\Theta^a\Theta^b},s}_L &= \frac{1}{2L+1}\sum_{\ell\ell'} w^{\mathrm{L},\Theta^a}_\ell w^{\mathrm{G},\Theta^b\Theta^a}_{\ell'} W^{\phi,0}_{\ell L\ell'} \frac{b_\ell b_{\ell'}}{2} \left[W^{\epsilon,0}_{\ell L\ell'} + W^{\epsilon,0}_{\ell' L\ell}\right] \\ &= \Sigma^{(0),\phi\epsilon}_L \left[bw^{\mathrm{L},\Theta^a}, bw^{\mathrm{G},\Theta^b\Theta^a}\right] + \Gamma^{(0),\phi\epsilon}_L \left[bw^{\mathrm{L},\Theta^a}, bw^{\mathrm{G},\Theta^b\Theta^a}\right] \,. \end{aligned} \tag{268}$$

$$\begin{aligned} R^{s^{\Theta^a\Theta^b},\phi}_L &= \frac{1}{2L+1}\sum_{\ell\ell'} \frac{1}{2} w^{\mathrm{L},\Theta^a}_\ell w^{\mathrm{L},\Theta^b}_{\ell'} b_\ell b_{\ell'} W^{\epsilon,0}_{\ell L\ell'} \left[W^{\phi,0}_{\ell L\ell'} C^{\Theta\Theta}_{\ell'} + W^{y,0}_{\ell' L\ell} C^{\Theta\Theta}_\ell\right] \\ &= \Sigma^{(0),\epsilon\phi}_L \left[bw^{\mathrm{L},\Theta^a}, bw^{\mathrm{L},\Theta^b} C^{\Theta\Theta}\right] + \Gamma^{(0),\epsilon\phi}_L \left[bw^{\mathrm{L},\Theta^a} C^{\Theta\Theta}, bw^{\mathrm{L},\Theta^b}\right] \\ &= \Sigma^{(0),\phi\epsilon}_L \left[bw^{\mathrm{L},\Theta^a}, bw^{\mathrm{L},\Theta^b} C^{\Theta\Theta}\right] + \Gamma^{(0),\phi\epsilon}_L \left[bw^{\mathrm{L},\Theta^b}, bw^{\mathrm{L},\Theta^a} C^{\Theta\Theta}\right] \,. \end{aligned} \tag{269}$$

For the asymmetric case, the normalization does not equal to the noise spectrum even in the idealistic case due to the correlation of the two anisotropies analogues to the ΘE estimator. The noise spectrum between $\widehat{x}^{\Theta\Theta}$ and $\widehat{y}^{\Theta\Theta}$ for the asymmetric estimators is given by

$$\begin{aligned} \frac{N^{x^{\Theta^a\Theta^b}y^{\Theta^{a'}\Theta^{b'}}}_L}{A^{x^{\Theta^a\Theta^b}}_L A^{y^{\Theta^{a'}\Theta^{b'}}}_L} &= \frac{1}{2L+1}\sum_{\ell\ell'} w^{\mathrm{L},\Theta^a}_\ell w^{\mathrm{G},\Theta^b\Theta^a}_{\ell'} W^{x,0}_{\ell L\ell'} [w^{\mathrm{L},\Theta^{a'}}_\ell w^{\mathrm{G},\Theta^{b'}\Theta^{a'}}_{\ell'} W^{y,0}_{\ell L\ell'} \widehat{\mathcal{C}}^{\Theta^a\Theta^{a'}}_\ell \widehat{\mathcal{C}}^{\Theta^b\Theta^{b'}}_{\ell'} + pw^{\mathrm{L},\Theta^{a'}}_{\ell'} w^{\mathrm{G},\Theta^{b'}\Theta^{a'}}_\ell W^{y,0}_{\ell' L\ell} \widehat{\mathcal{C}}^{\Theta^a\Theta^{b'}}_\ell \widehat{\mathcal{C}}^{\Theta^{a'}\Theta^b}_{\ell'}] \\ &= \Sigma^{(0),xy}_L \left[w^{\mathrm{L},\Theta^a} w^{\mathrm{L},\Theta^{a'}} \widehat{\mathcal{C}}^{\Theta^a\Theta^{a'}}, w^{\mathrm{G},\Theta^b\Theta^a} w^{\mathrm{G},\Theta^{b'}\Theta^{a'}} \widehat{\mathcal{C}}^{\Theta^b\Theta^{b'}}\right] + p\Gamma^{(0),xy}_L \left[w^{\mathrm{L},\Theta^a} w^{\mathrm{G},\Theta^{b'}\Theta^{a'}} \widehat{\mathcal{C}}^{\Theta^a\Theta^{b'}}, w^{\mathrm{L},\Theta^{a'}} w^{\mathrm{G},\Theta^b\Theta^a} \widehat{\mathcal{C}}^{\Theta^{a'}\Theta^b}\right] \,. \end{aligned} \tag{270}$$

6 Explicit Kernel Functions

Here we consider expression for the Kernel functions in terms of the Wigner d-functions. In the following calculations, we frequently use [15]

$$\int_{-1}^{1}\mathrm{d}\mu\ d^{\ell_1}_{s_1,s_1'}(\beta)d^{\ell_2}_{s_2,s_2'}(\beta)d^{\ell_3}_{s_3,s_3'}(\beta) = 2\begin{pmatrix}\ell_1 & \ell_2 & \ell_3\\ s_1 & s_2 & s_3\end{pmatrix}\begin{pmatrix}\ell_1 & \ell_2 & \ell_3\\ s_1' & s_2' & s_3'\end{pmatrix}\,, \tag{271}$$

with $s_1+s_2+s_3=s_1'+s_2'+s_3'=0$ and $\mu=\cos\beta$, and the symmetric property:

$$d^\ell_{mm'}(\beta) = (-1)^{m-m'}d^\ell_{-m,-m'}(\beta) = (-1)^{m-m'}d^\ell_{m'm}(\beta) \tag{272}$$

$$d^\ell_{mm'}(\beta) = (-1)^{\ell+m}d^\ell_{m,-m'}(\pi-\beta)\,. \tag{273}$$

Note that

$$(-1)^{\ell_1+\ell_2+\ell_3}\int_{-1}^{1}\mathrm{d}\mu\ d^{\ell_1}_{s_1,s_1'}d^{\ell_2}_{s_2,s_2'}d^{\ell_3}_{s_3,s_3'} = \int_{-1}^{1}\mathrm{d}\mu\ d^{\ell_1}_{s_1,-s_1'}d^{\ell_2}_{s_2,-s_2'}d^{\ell_3}_{s_3,-s_3'}\,. \tag{274}$$

We also define

$$X^{p\dots q} = (\sqrt{2}a^p_\ell)\dots(\sqrt{2}a^q_\ell)X_\ell\,. \tag{275}$$

and

$$\xi^A_{mm'} = \sum_\ell \frac{2\ell+1}{4\pi}A_\ell d^\ell_{mm'}\,. \tag{276}$$

For lensing, we obtain

$$p_x = c_x^2\,. \tag{277}$$

6.1 Kernel Functions: Lensing

For lensing fields, $x=\phi$ or ϖ, we obtain

$$\begin{aligned}
\Sigma^{(0),x}_L[A,B] &= \frac{1}{2L+1}\sum_{\ell\ell'}|W^{x,0}_{\ell L\ell'}|^2A_\ell B_{\ell'} \\
&= \sum_{\ell\ell'}4\pi\frac{2\ell+1}{4\pi}A_\ell\frac{2\ell'+1}{4\pi}B_{\ell'}\frac{L(L+1)}{2}\frac{\ell'(\ell'+1)}{2}[1+c_x^2(-1)^{\ell+L+\ell'}]2\begin{pmatrix}\ell & L & \ell'\\ 0 & 1 & -1\end{pmatrix}^2 \\
&= \int_{-1}^{1}\mathrm{d}\mu\ \pi L(L+1)\sum_{\ell\ell'}\frac{2\ell+1}{4\pi}A_\ell\frac{2\ell'+1}{4\pi}B_{\ell'}\ell'(\ell'+1)[d^\ell_{00}d^L_{11}d^{\ell'}_{11}+c_x^2d^\ell_{00}d^L_{1,-1}d^{\ell'}_{1,-1}] \\
&= \int_{-1}^{1}\mathrm{d}\mu\ \pi L(L+1)\{\xi^A_{00}\xi^{B^{00}}_{11}d^L_{11}+c_x^2\xi^A_{00}\xi^{B^{00}}_{1,-1}d^L_{1,-1}\}\,.
\end{aligned} \tag{278}$$

and

$$\begin{aligned}
\Gamma^{(0),x}_L[A,B] &= \frac{1}{2L+1}\sum_{\ell\ell'}(W^{x,0}_{\ell L\ell'})^*W^{x,0}_{\ell'L\ell}A_\ell B_{\ell'} \\
&= \sum_{\ell\ell'}2\pi L(L+1)\frac{2\ell+1}{4\pi}A_\ell\frac{2\ell'+1}{4\pi}B_{\ell'}a^0_\ell a^0_{\ell'}[1+c_x^2(-1)^{\ell+L+\ell'}]2\begin{pmatrix}\ell & L & \ell'\\ 0 & 1 & -1\end{pmatrix}\begin{pmatrix}\ell' & L & \ell\\ 0 & 1 & -1\end{pmatrix} \\
&= \sum_{\ell\ell'}\pi L(L+1)\frac{2\ell+1}{4\pi}A^0_\ell\frac{2\ell'+1}{4\pi}B^0_{\ell'}[1+c_x^2(-1)^{\ell+L+\ell'}]2\begin{pmatrix}\ell & L & \ell'\\ 0 & 1 & -1\end{pmatrix}\begin{pmatrix}\ell & L & \ell'\\ 1 & -1 & 0\end{pmatrix} \\
&= \int_{-1}^{1}\mathrm{d}\mu\ \pi L(L+1)\sum_{\ell\ell'}\frac{2\ell+1}{4\pi}A^0_\ell\frac{2\ell'+1}{4\pi}B^0_{\ell'}[d^\ell_{01}d^L_{1,-1}d^{\ell'}_{-1,0}+c_x^2d^\ell_{0,-1}d^L_{11}d^{\ell'}_{-1,0}] \\
&= -\int_{-1}^{1}\mathrm{d}\mu\ \pi L(L+1)\{\xi^{A^0}_{01}\xi^{B^0}_{0,-1}d^L_{1,-1}+c_x^2\xi^{A^0}_{01}\xi^{B^0}_{01}d^L_{11}\}\,.
\end{aligned} \tag{279}$$

Denoting $p=\pm$ and $x=\phi,\varpi$, we rewrite the kernel for polarization as

$$
\begin{aligned}
\Sigma_L^{(p),x}[A,B] &= \frac{1}{2L+1}\sum_{\ell\ell'}|W^{x,p}_{\ell L\ell'}|^2A_\ell B_{\ell'} \\
&= \frac{\pi}{2}L(L+1)\sum_{\ell\ell'}\frac{2\ell+1}{4\pi}A_\ell\frac{2\ell'+1}{4\pi}B_{\ell'}2[1+pc_x^2(-1)^{\ell+L+\ell'}]\left[a^+_{\ell'}\begin{pmatrix}\ell & L & \ell'\\ 2 & 1 & -3\end{pmatrix}+c_x^2a^-_{\ell'}\begin{pmatrix}\ell & L & \ell'\\ 2 & -1 & -1\end{pmatrix}\right]^2 \\
&= \frac{\pi}{2}L(L+1)\sum_{\ell\ell'}\frac{2\ell+1}{4\pi}A_\ell\frac{2\ell'+1}{4\pi}B_{\ell'}[1+pc_x^2(-1)^{\ell+L+\ell'}] \\
&\quad\times 2\left[(a^+_{\ell'})^2\begin{pmatrix}\ell & L & \ell'\\ 2 & 1 & -3\end{pmatrix}^2+(a^-_{\ell'})^2\begin{pmatrix}\ell & L & \ell'\\ 2 & -1 & -1\end{pmatrix}^2+2c_x^2a^+_{\ell'}a^-_{\ell'}\begin{pmatrix}\ell & L & \ell'\\ 2 & -1 & -1\end{pmatrix}\begin{pmatrix}\ell & L & \ell'\\ 2 & 1 & -3\end{pmatrix}\right] \\
&= \frac{\pi}{2}\int_{-1}^{1}\mathrm{d}\mu\; L(L+1)\sum_{\ell\ell'}\frac{2\ell+1}{4\pi}A_\ell\frac{2\ell'+1}{4\pi}B_{\ell'}[(a^+_{\ell'})^2d^\ell_{22}d^L_{11}d^{\ell'}_{33}+(a^-_{\ell'})^2d^\ell_{22}d^L_{11}d^{\ell'}_{11}+2c_x^2a^+_{\ell'}a^-_{\ell'}d^\ell_{22}d^L_{1,-1}d^{\ell'}_{13} \\
&\qquad +pc_x^2(a^+_{\ell'})^2d^\ell_{2,-2}d^L_{1,-1}d^{\ell'}_{3,-3}+pc_x^2(a^-_{\ell'})^2d^\ell_{2,-2}d^L_{1,-1}d^{\ell'}_{1,-1}+2pa^+_{\ell'}a^-_{\ell'}d^\ell_{2,-2}d^L_{11}d^{\ell'}_{1,-3}] \\
&= \int_{-1}^{1}\mathrm{d}\mu\;\frac{\pi}{4}L(L+1)[(\xi^A_{22}\xi^{B^{++}}_{33}+\xi^A_{22}\xi^{B^{--}}_{11}+2p\xi^A_{2,-2}\xi^{B^{+-}}_{3,-1})d^L_{11} \\
&\qquad +c_x^2(p\xi^A_{2,-2}\xi^{B^{++}}_{3,-3}+p\xi^A_{2,-2}\xi^{B^{--}}_{1,-1}+2\xi^A_{22}\xi^{B^{+-}}_{31})d^L_{1,-1}]\,,
\end{aligned}
\tag{280}
$$

and

$$
\begin{aligned}
\Gamma_L^{(p),x}[A,B] &= \frac{1}{2L+1}\sum_{\ell\ell'}(W^{x,p}_{\ell L\ell'})^*W^{x,p}_{\ell' L\ell}A_\ell B_{\ell'} \\
&= \frac{\pi}{2}L(L+1)\sum_{\ell\ell'}\frac{2\ell+1}{4\pi}A_\ell\frac{2\ell'+1}{4\pi}B_{\ell'}2[1+pc_x^2(-1)^{\ell+L+\ell'}] \\
&\quad\times\left[a^+_{\ell'}\begin{pmatrix}\ell & L & \ell'\\ 2 & 1 & -3\end{pmatrix}+c_x^2a^-_{\ell'}\begin{pmatrix}\ell & L & \ell'\\ 2 & -1 & -1\end{pmatrix}\right]\left[a^+_{\ell}\begin{pmatrix}\ell' & L & \ell\\ 2 & 1 & -3\end{pmatrix}+c_x^2a^-_{\ell}\begin{pmatrix}\ell' & L & \ell\\ 2 & -1 & -1\end{pmatrix}\right] \\
&= \frac{\pi}{2}L(L+1)\sum_{\ell\ell'}\frac{2\ell+1}{4\pi}A_\ell\frac{2\ell'+1}{4\pi}B_{\ell'}2[(-1)^{\ell+L+\ell'}+pc_x^2] \\
&\quad\times\left[a^+_{\ell'}\begin{pmatrix}\ell & L & \ell'\\ 2 & 1 & -3\end{pmatrix}+c_x^2a^-_{\ell'}\begin{pmatrix}\ell & L & \ell'\\ 2 & -1 & -1\end{pmatrix}\right]\left[a^+_{\ell}\begin{pmatrix}\ell & L & \ell'\\ -3 & 1 & 2\end{pmatrix}+c_x^2a^-_{\ell}\begin{pmatrix}\ell & L & \ell'\\ -1 & -1 & 2\end{pmatrix}\right] \\
&= \int_{-1}^{1}\mathrm{d}\mu\;\frac{\pi}{2}L(L+1)\sum_{\ell\ell'}\frac{2\ell+1}{4\pi}A_\ell\frac{2\ell'+1}{4\pi}B_{\ell'} \\
&\quad\times[a^+_{\ell'}a^+_\ell d^\ell_{23}d^L_{1,-1}d^{\ell'}_{-3,-2}+c_x^2a^+_{\ell'}a^-_\ell d^\ell_{21}d^L_{11}d^{\ell'}_{-3,-2}+c_x^2a^-_{\ell'}a^+_\ell d^\ell_{23}d^L_{11}d^{\ell'}_{-1,-2}+a^-_{\ell'}a^-_\ell d^\ell_{21}d^L_{1,-1}d^{\ell'}_{-1,-2} \\
&\quad+p(c_x^2a^+_{\ell'}a^+_\ell d^\ell_{2,-3}d^L_{11}d^{\ell'}_{-3,2}+a^+_{\ell'}a^-_\ell d^\ell_{2,-1}d^L_{1,-1}d^{\ell'}_{-3,2}+a^-_{\ell'}a^+_\ell d^\ell_{2,-3}d^L_{1,-1}d^{\ell'}_{-1,2}+c_x^2a^-_{\ell'}a^-_\ell d^\ell_{2,-1}d^L_{11}d^{\ell'}_{-1,2})] \\
&= \int_{-1}^{1}\mathrm{d}\mu\;\frac{\pi}{4}L(L+1)[-(\xi^{A^-}_{21}\xi^{B^+}_{32}+\xi^{A^+}_{32}\xi^{B^-}_{21}+p\xi^{A^+}_{3,-2}\xi^{B^+}_{3,-2}+p\xi^{A^-}_{2,-1}\xi^{B^-}_{2,-1})c_x^2d^L_{11} \\
&\quad+(\xi^{A^+}_{32}\xi^{B^+}_{32}+\xi^{A^-}_{21}\xi^{B^-}_{21}-p\xi^{A^-}_{2,-1}\xi^{B^+}_{3,-2}-p\xi^{A^+}_{3,-2}\xi^{B^-}_{2,-1})d^L_{1,-1}]\,.
\end{aligned}
\tag{281}
$$

The other kernels are given by

$$\begin{aligned}
\Sigma_L^{(\times),x}[A,B] &= \frac{1}{2L+1}\sum_{\ell\ell'}(W^{x,0}_{\ell L\ell'})^* W^{x,+}_{\ell L\ell'} A_\ell B_{\ell'} \\
&= \pi L(L+1)\sum_{\ell\ell'}\frac{2\ell+1}{4\pi}A_\ell\frac{2\ell'+1}{4\pi}B_{\ell'}a^0_{\ell'}2[1+c_x^2(-1)^{\ell+L+\ell'}] \\
&\quad\times\begin{pmatrix}\ell & L & \ell' \\ 0 & 1 & -1\end{pmatrix}\left[a^+_{\ell'}\begin{pmatrix}\ell & L & \ell' \\ 2 & 1 & -3\end{pmatrix}+c_x^2 a^-_{\ell'}\begin{pmatrix}\ell & L & \ell' \\ 2 & -1 & -1\end{pmatrix}\right] \\
&= \int_{-1}^1 \mathrm{d}\mu\ \pi L(L+1)\sum_{\ell\ell'}\frac{2\ell+1}{4\pi}A_\ell\frac{2\ell'+1}{4\pi}B_{\ell'}a^0_{\ell'} \\
&\quad\times\left(a^+_{\ell'}d^\ell_{20}d^L_{11}d^{\ell'}_{31}+c_x^2a^-_{\ell'}d^\ell_{20}d^L_{1,-1}d^{\ell'}_{11}+c_x^2a^+_{\ell'}d^\ell_{0,-2}d^L_{1,-1}d^{\ell'}_{-1,3}+a^-_{\ell'}d^\ell_{0,-2}d^L_{11}d^{\ell'}_{-1,1}\right) \\
&= \int_{-1}^1 \mathrm{d}\mu\ \frac{\pi}{2}L(L+1)\xi^A_{20}\left[(\xi^{B^{0+}}_{31}+\xi^{B^{0-}}_{1,-1})d^L_{11}+c_x^2(\xi^{B^{0-}}_{11}+\xi^{B^{0+}}_{3,-1})d^L_{1,-1}\right]\,, \qquad (282)
\end{aligned}$$

and

$$\begin{aligned}
\Gamma_L^{(\times),x}[A,B] &= \frac{1}{2L+1}\sum_{\ell\ell'}(W^{x,0}_{\ell L\ell'})^* W^{x,+}_{\ell' L\ell} A_\ell B_{\ell'} \\
&= \pi L(L+1)\sum_{\ell\ell'}\frac{2\ell+1}{4\pi}A_\ell\frac{2\ell'+1}{4\pi}B_{\ell'}a^0_{\ell'}2[1+c_x^2(-1)^{\ell+L+\ell'}] \\
&\quad\times\begin{pmatrix}\ell & L & \ell' \\ 0 & 1 & -1\end{pmatrix}\left[a^+_{\ell}\begin{pmatrix}\ell' & L & \ell \\ 2 & 1 & -3\end{pmatrix}+c_x^2 a^-_{\ell}\begin{pmatrix}\ell' & L & \ell \\ 2 & -1 & -1\end{pmatrix}\right] \\
&= \pi L(L+1)\sum_{\ell\ell'}\frac{2\ell+1}{4\pi}A_\ell\frac{2\ell'+1}{4\pi}B_{\ell'}a^0_{\ell'}2[1+c_x^2(-1)^{\ell+L+\ell'}] \\
&\quad\times\begin{pmatrix}\ell & L & \ell' \\ 0 & 1 & -1\end{pmatrix}\left[a^+_{\ell}\begin{pmatrix}\ell & L & \ell' \\ 3 & -1 & -2\end{pmatrix}+c_x^2 a^-_{\ell}\begin{pmatrix}\ell & L & \ell' \\ 1 & 1 & -2\end{pmatrix}\right] \\
&= \int_{-1}^1 \mathrm{d}\mu\ \pi L(L+1)\sum_{\ell\ell'}\frac{2\ell+1}{4\pi}A_\ell\frac{2\ell'+1}{4\pi}B_{\ell'}a^0_{\ell'} \\
&\quad\times\left(a^+_{\ell}d^\ell_{03}d^L_{1,-1}d^{\ell'}_{-1,-2}+c_x^2a^-_{\ell}d^\ell_{01}d^L_{11}d^{\ell'}_{-1,-2}+c_x^2a^+_{\ell}d^\ell_{0,-3}d^L_{11}d^{\ell'}_{-1,2}+a^-_{\ell}d^\ell_{0,-1}d^L_{1,-1}d^{\ell'}_{-1,2}\right) \\
&= -\int_{-1}^1 \mathrm{d}\mu\ \frac{\pi}{2}L(L+1)\left[(\xi^{A^+}_{30}\xi^{B^0}_{21}+\xi^{A^-}_{10}\xi^{B^0}_{2,-1})d^L_{1,-1}+c_x^2(\xi^{A^-}_{10}\xi^{B^0}_{21}+\xi^{A^+}_{30}\xi^{B^0}_{2,-1})d^L_{11}\right]\,. \qquad (283)
\end{aligned}$$

6.2 Kernel Functions: Amplitude

Here we consider $x=\epsilon$. For $s=0$, we obtain

$$\begin{aligned}
\Sigma_L^{0,\epsilon}[A,B] &= \frac{1}{2L+1}\sum_{\ell\ell'}A_\ell B_{\ell'}p^+_{\ell L\ell'}(\gamma_{\ell L\ell'})^2\begin{pmatrix}\ell & L & \ell' \\ 0 & 0 & 0\end{pmatrix}^2 \\
&= \int_{-1}^1 \mathrm{d}\mu\ \pi\sum_{\ell\ell'}\frac{2\ell+1}{4\pi}A_\ell\frac{2\ell'+1}{4\pi}B_{\ell'}2d^\ell_{00}d^L_{00}d^{\ell'}_{00} \\
&= \int_{-1}^1 \mathrm{d}\mu\ 2\pi\xi^A_{00}\xi^B_{00}d^L_{00}\,. \qquad (284)
\end{aligned}$$

Using the property of the distortion function, we find

$$\Gamma_L^{0,\epsilon}[A,B] = \Sigma_L^{0,\epsilon}[A,B]\,. \qquad (285)$$

For $s = \pm$, the weight function is given by

$$\begin{aligned}\Sigma_L^{(\pm),\epsilon}[A,B] &= \pi\sum_{\ell\ell'}\frac{2\ell+1}{4\pi}A_\ell\frac{2\ell'+1}{4\pi}B_{\ell'}2[1\pm(-1)^{\ell+L+\ell'}]\begin{pmatrix}\ell & L & \ell'\\ 2 & 0 & -2\end{pmatrix}^2 \\ &= \int_{-1}^{1}\mathrm{d}\mu\;\pi\sum_{\ell\ell'}\frac{2\ell+1}{4\pi}A_\ell\frac{2\ell'+1}{4\pi}B_{\ell'}(d^\ell_{22}d^L_{00}d^{\ell'}_{22}\pm d^\ell_{2,-2}d^L_{00}d^{\ell'}_{2,-2}) \\ &= \int_{-1}^{1}\mathrm{d}\mu\;\pi(\xi^A_{22}\xi^B_{22}\pm\xi^A_{2,-2}\xi^B_{2,-2})d^L_{00}\,. \end{aligned} \tag{286}$$

Using the property of the distortion function, we find

$$\Gamma_L^{(\pm),\epsilon}[A,B] = \Sigma_L^{(\pm),\epsilon}[A,B]\,. \tag{287}$$

For ΘE,

$$\begin{aligned}\Sigma_L^{(\times),\epsilon}[A,B] &= \frac{1}{2L+1}\sum_{\ell\ell'}(W^{\epsilon,0}_{\ell L\ell'})^*W^{\epsilon,+}_{\ell L\ell'}A_\ell B_{\ell'} \\ &= \frac{1}{2L+1}\sum_{\ell\ell'}\gamma^2_{\ell L\ell'}\frac{1+(-1)^{\ell+L+\ell'}}{2}\begin{pmatrix}\ell & L & \ell'\\ 0 & 0 & 0\end{pmatrix}\begin{pmatrix}\ell & L & \ell'\\ 2 & 0 & -2\end{pmatrix}A_\ell B_{\ell'} \\ &= 2\pi\sum_{\ell\ell'}\frac{(2\ell+1)}{4\pi}A_\ell\frac{(2\ell'+1)}{4\pi}B_{\ell'}\left[\begin{pmatrix}\ell & L & \ell'\\ 2 & 0 & -2\end{pmatrix}+\begin{pmatrix}\ell & L & \ell'\\ -2 & 0 & 2\end{pmatrix}\right]\begin{pmatrix}\ell & L & \ell'\\ 0 & 0 & 0\end{pmatrix} \\ &= \pi\sum_{\ell\ell'}\frac{(2\ell+1)}{4\pi}A_\ell\frac{(2\ell'+1)}{4\pi}B_{\ell'}\int_{-1}^{1}\mathrm{d}\mu\;(d^\ell_{20}d^L_{00}d^{\ell'}_{-2,0}+d^\ell_{-2,0}d^L_{00}d^{\ell'}_{20}) \\ &= 2\pi\int_{-1}^{1}\mathrm{d}\mu\;\zeta^A_{20}\zeta^B_{20}d^L_{00}\,, \end{aligned} \tag{288}$$

where we use $d^\ell_{-2,0} = d^\ell_{20}$. Using the property of the weight function, we obtain

$$\Gamma_L^{(\times),\epsilon}[A,B] = \Sigma_L^{(\times),\epsilon}[A,B]\,. \tag{289}$$

6.3 Kernel Functions: Rotation

The kernel functions for $x = \alpha$ is easily obtained from that for $x = \epsilon$. Using the property of the distortion function, we find that

$$\Sigma_L^{(\pm),\alpha}[A,B] = 4\Sigma_L^{(\mp),\epsilon}[A,B]\,, \tag{290}$$
$$\Gamma_L^{(\pm),\alpha}[A,B] = 4\Gamma_L^{(\mp),\epsilon}[A,B]\,, \tag{291}$$
$$\Sigma_L^{0,\alpha}[A,B] = 0\,, \tag{292}$$
$$\Gamma_L^{0,\alpha}[A,B] = 0\,, \tag{293}$$
$$\Sigma_L^{\times,\alpha}[A,B] = 0\,, \tag{294}$$
$$\Gamma_L^{\times,\alpha}[A,B] = 0\,. \tag{295}$$

6.4 Response function

6.4.1 ϕ and ϵ

The lensing potential and amplitude modulation are both even. We then need to compute

$$\begin{aligned}W^{\phi,0}_{\ell L\ell'}W^{\epsilon,0}_{\ell L\ell'} &= -2(p^+_{\ell L\ell'})^2(\gamma_{\ell L\ell'})^2a^0_La^0_{\ell'}\begin{pmatrix}\ell & L & \ell'\\ 0 & 1 & -1\end{pmatrix}\begin{pmatrix}\ell & L & \ell'\\ 0 & 0 & 0\end{pmatrix} \\ &= -\int_{-1}^{1}\mathrm{d}\mu\;(\gamma_{\ell L\ell'})^2a^0_La^0_{\ell'}d^\ell_{00}d^L_{10}d^{\ell'}_{-1,0}\,. \end{aligned} \tag{296}$$

Then we obtain

$$\Sigma_L^{0,\phi\epsilon}[A,B] = \int_{-1}^{1}\mathrm{d}\mu\ 2\pi\sqrt{L(L+1)}d^L_{10}\xi^A_{00}\xi^{B^0}_{10}\ , \tag{297}$$

and

$$\Gamma_L^{0,\phi\epsilon}[A,B] = \Sigma_L^{0,\phi\epsilon}[A,B]\,. \tag{298}$$

For polarization,

$$W^{\phi,\pm}_{\ell L\ell'}W^{\epsilon,\pm}_{\ell L\ell'} = -(\zeta^\pm)^2(p^\pm_{\ell L\ell'})^2(\gamma_{\ell L\ell'})^2a^0_L\left[a^+_{\ell'}\begin{pmatrix}\ell & L & \ell' \\ 2 & 1 & -3\end{pmatrix} + a^-_{\ell'}\begin{pmatrix}\ell & L & \ell' \\ 2 & -1 & -1\end{pmatrix}\right]\begin{pmatrix}\ell & L & \ell' \\ 2 & 0 & -2\end{pmatrix}\,. \tag{299}$$

We obtain

$$\begin{aligned}W^{\phi,\pm}_{\ell L\ell'}W^{\epsilon,\pm}_{\ell L\ell'} &= \mp p^\pm_{\ell L\ell'}(\gamma_{\ell L\ell'})^2a^0_L\left[a^+_{\ell'}\begin{pmatrix}\ell & L & \ell' \\ 2 & 1 & -3\end{pmatrix}\begin{pmatrix}\ell & L & \ell' \\ 2 & 0 & -2\end{pmatrix} + a^-_{\ell'}\begin{pmatrix}\ell & L & \ell' \\ 2 & -1 & -1\end{pmatrix}\begin{pmatrix}\ell & L & \ell' \\ 2 & 0 & -2\end{pmatrix}\right] \\ &= \int_{-1}^{1}\mathrm{d}\mu\ \frac{\mp1}{4}(\gamma_{\ell L\ell'})^2a^0_L\left[a^+_{\ell'}d^\ell_{22}d^L_{-1,0}d^{\ell'}_{32} + a^-_{\ell'}d^\ell_{22}d^L_{10}d^{\ell'}_{12} \pm a^+_{\ell'}d^\ell_{2,-2}d^L_{-1,0}d^{\ell'}_{3,-2} \pm a^-_{\ell'}d^\ell_{2,-2}d^L_{10}d^{\ell'}_{1,-2}\right]\ ,\end{aligned} \tag{300}$$

The kernel function is given by

$$\Sigma_L^{(\pm),\phi\epsilon}[A,B] = \mp\int_{-1}^{1}\mathrm{d}\mu\ \frac{\pi}{2}\sqrt{L(L+1)}d^L_{10}\left(\xi^A_{22}\xi^{B^+}_{32} - \xi^A_{22}\xi^{B^-}_{21} \pm \xi^A_{2,-2}\xi^{B^+}_{3,-2} \pm \xi^A_{2,-2}\xi^{B^-}_{2,-1}\right)\ , \tag{301}$$

and, using the property of the weight function, we find

$$\Gamma_L^{(\pm),\phi\epsilon}[A,B] = \Sigma_L^{(\pm),\phi\epsilon}[A,B]\,. \tag{302}$$

The cross kernels are:

$$\begin{aligned}W^{\phi,0}_{\ell L\ell'}W^{\epsilon,+}_{\ell L\ell'} &= a^0_La^0_{\ell'}p^2_{\ell L\ell'}(\gamma_{\ell L\ell'})^2 2\begin{pmatrix}\ell & L & \ell' \\ 0 & 1 & 1\end{pmatrix}\begin{pmatrix}\ell & L & \ell' \\ 2 & 0 & -2\end{pmatrix} = W^{\phi,0}_{\ell L\ell'}W^{\epsilon,+}_{\ell' L\ell} \\ &= \int_{-1}^{1}\mathrm{d}\mu\ (\gamma_{\ell L\ell'})^2a^0_La^0_{\ell'}p_{\ell L\ell'}d^\ell_{02}d^L_{10}d^{\ell'}_{1,-2} \\ &= \int_{-1}^{1}\mathrm{d}\mu\ (\gamma_{\ell L\ell'})^2a^0_La^0_{\ell'}\frac{1}{2}(d^\ell_{02}d^{\ell'}_{1,-2} + d^\ell_{0,-2}d^{\ell'}_{12})d^L_{10} \\ &= \int_{-1}^{1}\mathrm{d}\mu\ (\gamma_{\ell L\ell'})^2a^0_La^0_{\ell'}\frac{1}{2}d^\ell_{20}(d^{\ell'}_{2,-1} - d^{\ell'}_{21})d^L_{10}\,,\end{aligned} \tag{303}$$

and

$$\begin{aligned}W^{\epsilon,0}_{\ell L\ell'}W^{\phi,+}_{\ell L\ell'} &= -a^0_Lp_{\ell L\ell'}(\gamma_{\ell L\ell'})^2\begin{pmatrix}\ell & L & \ell' \\ 0 & 0 & 0\end{pmatrix}\left[a^+_{\ell'}\begin{pmatrix}\ell & L & \ell' \\ 2 & 1 & -3\end{pmatrix} + a^-_{\ell'}\begin{pmatrix}\ell & L & \ell' \\ 2 & -1 & -1\end{pmatrix}\right] \\ &= -\int_{-1}^{1}\mathrm{d}\mu\ \frac{1}{2}a^0_L(\gamma_{\ell L\ell'})^2d^\ell_{20}(a^+_{\ell'}d^{\ell'}_{-3,0} + a^-_{\ell'}d^{\ell'}_{-1,0})d^L_{-1,0} \\ &= \int_{-1}^{1}\mathrm{d}\mu\ \frac{1}{2}a^0_L(\gamma_{\ell L\ell'})^2d^\ell_{20}(a^+_{\ell'}d^{\ell'}_{30} - a^-_{\ell'}d^{\ell'}_{10})d^L_{10}\,.\end{aligned} \tag{304}$$

Then, we obtain:

$$\Sigma_L^{(\times),\phi\epsilon}[A,B] = \int_{-1}^{1}\mathrm{d}\mu\ \pi\sqrt{L(L+1)}\zeta^A_{20}(\zeta^{B^0}_{2,-1} - \zeta^{B^0}_{21})d^L_{10}\,, \tag{305}$$

$$\Gamma_L^{(\times),\phi\epsilon}[A,B] = \Sigma_L^{(\times),\phi\epsilon}[A,B]\,, \tag{306}$$

$$\Sigma_L^{(\times),\epsilon\phi}[A,B] = \int_{-1}^{1}\mathrm{d}\mu\ \pi\sqrt{L(L+1)}\zeta^A_{20}(\zeta^{B^+}_{30} - \zeta^{B^-}_{10})d^L_{10}\,, \tag{307}$$

$$\Gamma_L^{(\times),\epsilon\phi}[A,B] = \Sigma_L^{(\times),\epsilon\phi}[B,A]\,. \tag{308}$$

6.4.2 ϕ and s

The lensing potential and sources are both even. For s, the weight is obtained by replacing $C^{\Theta\Theta}$ in the numerator with $1/2$.

6.4.3 α and ϵ

The response of the quadratic EB estimator is given by

$$\begin{aligned}
[A_L^{\alpha\epsilon,(EB)}]^{-1} &= \frac{1}{2L+1}\sum_{\ell\ell'}\frac{(W^{\alpha,-}_{\ell L\ell'}C^{\rm BB}_{\ell'} - W^{\alpha,-}_{\ell' L\ell}C^{\rm EE}_{\ell})(W^{\epsilon,+}_{\ell L\ell'}C^{\rm EB}_{\ell'} + W^{\epsilon,+}_{\ell' L\ell}C^{\rm EB}_{\ell})}{\widehat{C}^{\rm EE}_\ell\widehat{C}^{\rm BB}_{\ell'}} \\
&= \frac{-1}{2(2L+1)}\sum_{\ell\ell'}\frac{(W^{\alpha,-}_{\ell L\ell'}C^{\rm BB}_{\ell'} - W^{\alpha,-}_{\ell' L\ell}C^{\rm EE}_{\ell})(W^{\alpha,-}_{\ell L\ell'}C^{\rm EB}_{\ell'} + W^{\alpha,-}_{\ell' L\ell}C^{\rm EB}_{\ell})}{\widehat{C}^{\rm EE}_\ell\widehat{C}^{\rm BB}_{\ell'}} \\
&= -\frac{1}{2}\Sigma_L^{(-),\alpha}\left[\frac{1}{\widehat{C}^{\rm EE}},\frac{C^{\rm EB}C^{\rm BB}}{\widehat{C}^{\rm BB}}\right] + \frac{1}{2}\Gamma_L^{(-),\alpha}\left[\frac{C^{\rm EE}}{\widehat{C}^{\rm EE}},\frac{(C^{\rm EB}-C^{\rm BB})}{\widehat{C}^{\rm BB}}\right] + \frac{1}{2}\Sigma_L^{(-),\alpha}\left[\frac{1}{\widehat{C}^{\rm BB}},\frac{C^{\rm EB}C^{\rm EE}}{\widehat{C}^{\rm EE}}\right] ,
\end{aligned} \tag{309}$$

7 Bias-hardened quadratic estimator

7.1 Definition

Assuming that an estimator has several mean-fields, the expectation of the estimator becomes:

$$\langle\widehat{x}_{LM}\rangle_{\rm CMB} = \sum_{x'} R^{xx'}_L x'_{LM} \equiv \mathbf{R}_L \boldsymbol{x'}_{LM} \,, \tag{310}$$

where $R^{xx}_L = 1$ by definition and $R^{xx'}$ is the response function. We can construct a bias-hardened estimators as [17]:

$$\widehat{x}^{\rm BH}_{LM} \equiv \sum_{x'} [\mathbf{R}^{-1}]^{xx'}_L \widehat{x}'_{LM} \,, \tag{311}$$

which is insensitive to the source of mean-field bias:

$$\langle\widehat{x}^{\rm BH}_{LM}\rangle = x_{LM} \,. \tag{312}$$

For a given two estimators, the response function satisfies

$$\langle\widehat{x}_{LM}(\widehat{y}_{LM})^*\rangle|_{x,y=0} = A^{xx}_L A^{yy}_L \bar{R}^{xy}_L = A^{xx}_L R^{yx}_L = R^{xy}_L A^{yy}_L \,, \tag{313}$$

where $\bar{R}^{xy}_L$ is a symmetric unnormalized response. The above equation is rewritten as:

$$\langle\widehat{\boldsymbol{y}}_{LM}(\widehat{\boldsymbol{y}}_{LM})^\dagger\rangle|_{x,y=0} = \mathbf{R}_L \begin{pmatrix} A^{y_1y_1} & \cdots & 0 \\ \vdots & & \vdots \\ 0 & \cdots & A^{y_ny_n} \end{pmatrix} \,, \tag{314}$$

where the later matrix is diagonal.

7.2 Noise

The idealistic reconstruction noise is the diagonal elements of the following matrix:

$$\begin{aligned}\langle\widehat{\boldsymbol{x}}^{\rm BH}(\widehat{\boldsymbol{x}}^{\rm BH})^\dagger\rangle = \mathbf{R}^{-1}\langle\widehat{\boldsymbol{y}}\widehat{\boldsymbol{y}}^\dagger\rangle(\mathbf{R}^{-1})^T = \mathbf{R}^{-1}\mathbf{R} \begin{pmatrix} A^{y_1y_1} & \cdots & 0 \\ \vdots & & \vdots \\ 0 & \cdots & A^{y_ny_n} \end{pmatrix} (\mathbf{R}^{-1})^T \\ = \begin{pmatrix} A^{y_1y_1} & \cdots & 0 \\ \vdots & & \vdots \\ 0 & \cdots & A^{y_ny_n} \end{pmatrix} (\mathbf{R}^{-1})^T\end{aligned} \tag{315}$$

Thus, we obtain

$$A^{xx,({\rm BH})} = A^{xx}\{\mathbf{R}^{-1}\}_{xx} \tag{316}$$

For two estimator case, the above equation becomes

$$A^{xx,({\rm BH})} = \frac{A^{xx}}{1-R^{xy}R^{yx}} = \frac{A^{xx}}{1-A^{xx}A^{yy}(\bar{R}^{xy})^2} \,. \tag{317}$$

7.3 Example

7.3.1 Two estimators

$$\langle\widehat{x}^{\Theta\Theta}_{LM}\rangle_{\rm CMB} = x_{LM} + R^{xy}_L y_{LM} \,, \tag{318}$$

$$\langle\widehat{y}^{\Theta\Theta}_{LM}\rangle_{\rm CMB} = y_{LM} + R^{yx}_L x_{LM} \,, \tag{319}$$

where the response functions are given by:

$$R^{xy}_L = \frac{1}{2L+1}\sum_{\ell\ell'} \frac{W^{x,0}_{\ell L\ell'}C^{\Theta\Theta}_{\ell'} + pW^{x,0}_{\ell' L\ell}C^{\Theta\Theta}_{\ell}}{2\widehat{C}^{\Theta\Theta}_{\ell}\widehat{C}^{\Theta\Theta}_{\ell'}} \left[W^{y,0}_{\ell L\ell'}C^{\Theta\Theta}_{\ell'} + pW^{y,0}_{\ell' L\ell}C^{\Theta\Theta}_{\ell}\right] \,, \tag{320}$$

8 Computing delensed CMB anisotropies

8.1 Linear template of lensing B modes

The gradient of lensing potential $\boldsymbol{\nabla}\phi$ is transformed as

$$\begin{aligned}\boldsymbol{\nabla}\phi &= \sum_{\ell m}\boldsymbol{\nabla}Y_{\ell m}\phi_{\ell m} = -\sum_{\ell m}\sqrt{\frac{\ell(\ell+1)}{2}}\phi_{\ell m}(Y^1_{\ell m}\boldsymbol{e}^* - Y^{-1}_{\ell m}\boldsymbol{e}) \\ &= (\phi^1_r + \mathrm{i}\phi^1_i)\boldsymbol{e}^* + (\phi^1_r - \mathrm{i}\phi^1_i)\boldsymbol{e}\,, \end{aligned} \tag{321}$$

where $\phi^1_{r,i}$ are obtained by spin-1 inverse harmonic transform of $\phi_{\ell m}\sqrt{\ell(\ell+1)/2}$. Similarly the gradient of polarization $\boldsymbol{\nabla}P^\pm = \boldsymbol{\nabla}(Q\pm\mathrm{i}U)$ is

$$\begin{aligned}\boldsymbol{\nabla}P^+ &= -\sum_{\ell m}E_{\ell m}\boldsymbol{\nabla}Y^2_{\ell m} \\ &= \sum_{\ell m}E_{\ell m}\left(\sqrt{\frac{(\ell-2)(\ell+3)}{2}}Y^3_{\ell m}\boldsymbol{e}^* - \sqrt{\frac{(\ell+2)(\ell-1)}{2}}Y^1_{\ell m}\boldsymbol{e}\right) \\ &= -(E^+_3 + \mathrm{i}E^-_3)\boldsymbol{e}^* + (E^+_1 + \mathrm{i}E^-_1)\boldsymbol{e}\,, \end{aligned} \tag{322}$$

$$\boldsymbol{\nabla}P^- = (\boldsymbol{\nabla}P^+)^* = (E^+_1 - \mathrm{i}E^-_1)\boldsymbol{e}^* - (E^+_3 - \mathrm{i}E^-_3)\boldsymbol{e}\,. \tag{323}$$

This leads to

$$\boldsymbol{\nabla}\phi\cdot\boldsymbol{\nabla}P^+ = -(E^+_3 + \mathrm{i}E^-_3)(\phi^1_r - \mathrm{i}\phi^1_i) + (E^+_1 + \mathrm{i}E^-_1)(\phi^1_r + \mathrm{i}\phi^1_i) \tag{324}$$

$$\boldsymbol{\nabla}\phi\cdot\boldsymbol{\nabla}P^- = (\boldsymbol{\nabla}\phi\cdot\boldsymbol{\nabla}P^+)^*\,. \tag{325}$$

The harmonic transform of the above quantity becomes the leading-order lensing contributions to E/B.

8.2 Linear template of curl-mode induced B modes

Similarly, from Eq. (22), in the case of curl mode, we obtain

$$(\star\boldsymbol{\nabla})\varpi = \mathrm{i}[(\varpi^1_r + \mathrm{i}\varpi^1_i)\boldsymbol{e}^* - (\varpi^1_r - \mathrm{i}\varpi^1_i)\boldsymbol{e}]\,, \tag{326}$$

where we define

$$\varpi^1_r + \mathrm{i}\varpi^1_i = \sum_{\ell m}\sqrt{\frac{\ell(\ell+1)}{2}}\varpi_{\ell m}Y^1_{\ell m}\,. \tag{327}$$

We then obtain

$$(\star\boldsymbol{\nabla})\varpi\cdot\boldsymbol{\nabla}P^+ = \mathrm{i}(E^+_3 + \mathrm{i}E^-_3)(\varpi^1_r - \mathrm{i}\varpi^1_i) + \mathrm{i}(E^+_1 + \mathrm{i}E^-_1)(\varpi^1_r + \mathrm{i}\varpi^1_i)\,, \tag{328}$$

$$(\star\boldsymbol{\nabla})\varpi\cdot\boldsymbol{\nabla}P^- = [(\star\boldsymbol{\nabla})\varpi\cdot\boldsymbol{\nabla}P^+]^*\,. \tag{329}$$

9 Optimal filtering

9.1 Background

The inverse variance Wiener filtering is defined as

$$\left[\mathbf{C}^{-1} + \sum_k \mathbf{A}_k^\dagger \mathbf{N}_k^{-1} \mathbf{A}_k\right] \boldsymbol{x} = \sum_k \mathbf{A}_k^\dagger \mathbf{N}^{-1} \boldsymbol{d}_k \,, \tag{330}$$

where k is the index of frequency channels and different maps (e.g. LAT and SAT for SO), $\mathbf{C}$ is the signal covariance matrix, $\mathbf{N}_k$ is the noise covariance matrix in pixel space, $\mathbf{A}_k$ is a matrix that transforms the harmonic coefficients to a map in pixel space including beam and pixel convolution. From the data, $\boldsymbol{d}_k$, we solve $\boldsymbol{x}$ which is an array of the harmonic coefficients. The above equation is rewritten by the following numerically convenient form:

$$\left[1 + \mathbf{C}^{1/2}\left(\sum_k \mathbf{Y}_k^\dagger \mathbf{N}_k^{-1} \mathbf{Y}_k\right)\mathbf{C}^{1/2}\right](\mathbf{C}^{-1/2}\boldsymbol{x}) = \mathbf{C}^{1/2} \sum_k \mathbf{Y}_k^\dagger \mathbf{N}_k^{-1} \boldsymbol{d}_k \,, \tag{331}$$

where $(\mathbf{C}^{1/2})^2 = \mathbf{C}$. Using the spherical harmonics, $Y_{\ell m}$, we define

$$\mathbf{Y}_k \boldsymbol{x} = \sum_\ell \sum_{m=-\ell}^{\ell} b_\ell^k x_{\ell m} Y_{\ell m}(\hat{\boldsymbol{n}}) \,. \tag{332}$$

Here, b_ℓ^k is the one dimensional beam and pixel function, and $\hat{\boldsymbol{n}}_i$ denotes pixel position. Similarly,

$$\mathbf{Y}_k^\dagger \boldsymbol{x} = b_\ell^k \int \mathrm{d}^2\hat{\boldsymbol{n}}\; x(\hat{\boldsymbol{n}}) Y_{\ell m}^*(\hat{\boldsymbol{n}}) \,. \tag{333}$$

The operation involving the noise covariance is then becomes

$$\{\mathbf{Y}_k^\dagger \mathbf{N}_k^{-1} \mathbf{Y}_k \boldsymbol{x}\}_{\ell' m'} = \int \mathrm{d}^2\hat{\boldsymbol{n}}_j\; b_{\ell'}^k Y_{\ell' m'}^*(\hat{\boldsymbol{n}}_j) \int \mathrm{d}^2\hat{\boldsymbol{n}}_i\; \mathbf{N}^{-1}(\hat{\boldsymbol{n}}_i, \hat{\boldsymbol{n}}_j) \sum_{\ell m} b_\ell^k Y_{\ell m}(\hat{\boldsymbol{n}}_i) x_{\ell m} \,. \tag{334}$$

If the noise covariance is diagonal in pixel space and the signal matrix is diagonal in harmonic space, the matrix multiplication to an array of the harmonic coefficients becomes very simple. The conjugate gradient decent in the code solves $\boldsymbol{v}$ which satisfies

$$\mathbf{A}\boldsymbol{v} = \boldsymbol{b} \,, \tag{335}$$

where

$$\mathbf{A} = \left[1 + \mathbf{C}^{1/2}\left(\sum_k \mathbf{Y}_k^\dagger \mathbf{N}_k^{-1} \mathbf{Y}_k\right)\mathbf{C}^{1/2}\right] , \tag{336}$$

$$\boldsymbol{b} = \mathbf{C}^{1/2} \sum_k \mathbf{Y}_k^\dagger \mathbf{N}_k^{-1} \boldsymbol{d}_k \,. \tag{337}$$

The solution, $\boldsymbol{v}$, is then transformed to $\boldsymbol{x}$.

9.2 Inverse noise covariance

If the noise covariance in pixel space is diagonal,

$$\{\mathbf{N}\}_{ij} \equiv \langle n(\hat{\boldsymbol{n}}_i) n(\hat{\boldsymbol{n}}_j)\rangle = \delta(\hat{\boldsymbol{n}}_i - \hat{\boldsymbol{n}}_j)\sigma^2(\hat{\boldsymbol{n}}_i) \,, \tag{338}$$

we obtain

$$\{\mathbf{Y}^\dagger\mathbf{N}^{-1}\mathbf{Y}\boldsymbol{x}\}_{\ell'm'} = \int \mathrm{d}^2\hat{\boldsymbol{n}}_j\ Y^*_{\ell'm'}(\hat{\boldsymbol{n}}_j) \int \mathrm{d}^2\hat{\boldsymbol{n}}_i\ \sigma^2(\hat{\boldsymbol{n}}_i)\delta(\hat{\boldsymbol{n}}_i - \hat{\boldsymbol{n}}_j)\sum_{\ell m} Y_{\ell m}(\hat{\boldsymbol{n}}_i)x_{\ell m}$$
$$= \int \mathrm{d}^2\hat{\boldsymbol{n}}_j\ Y^*_{\ell'm'}(\hat{\boldsymbol{n}}_j)\sigma^2(\hat{\boldsymbol{n}}_j)\sum_{\ell m} Y_{\ell m}(\hat{\boldsymbol{n}}_j)x_{\ell m}\,, \tag{339}$$

where we ignore signal and beam. This operation is very efficient.

For a white uniform noise with σ [μK'], the noise covariance in pixel space becomes

$$\{\mathbf{N}\}_{ij} = \delta(\hat{\boldsymbol{n}}_i - \hat{\boldsymbol{n}}_j)\left(\frac{\sigma}{T_{\rm CMB}}\frac{\pi}{10800}\right)^2 \equiv \delta(\hat{\boldsymbol{n}}_i - \hat{\boldsymbol{n}}_j)N^{\rm white}\,. \tag{340}$$

Then, the above filtering is equivalent to the usual diagonal filtering:

$$\{\mathbf{A}\}_{\ell m,\ell'm'} = \delta_{\ell\ell'}\delta_{mm'}\left[1 + \frac{C_\ell}{N_\ell}\right]\,, \tag{341}$$

$$\{\boldsymbol{b}\}_{\ell m} = \frac{C_\ell^{1/2}}{N_\ell}(s_{\ell m} + n^b_{\ell m})\,, \tag{342}$$

where C_ℓ is the beam-deconvolved signal spectrum, $N_\ell = N^{\rm white}/b_\ell^2$, $s_{\ell m}$ is the signal and $n^b_{\ell m} = n_{\ell m}/b_\ell$ is the noise divided by beam. Substituting the above equations into Eq. (335), we obtain

$$x_{\ell m} = C_\ell^{1/2}v_{\ell m} = \frac{C_\ell}{C_\ell + N_\ell}(s_{\ell m} + n^b_{\ell m})\,. \tag{343}$$

The noise variance from some simulated noise is given by

$$\{\mathbf{N}\}_{ij} = W(\hat{\boldsymbol{n}}_i)W(\hat{\boldsymbol{n}}_j)\sum_{\ell m\ell'm'} Y^*_{\ell m}(\hat{\boldsymbol{n}}_i)Y_{\ell'm'}(\hat{\boldsymbol{n}}_j)\langle n^*_{\ell m}n_{\ell'm'}\rangle\,, \tag{344}$$

where W represents inhomogeneities of scan. For a uniform noise with $\langle n^*_{\ell m}n_{\ell'm'}\rangle = \sigma_0^2\delta_{\ell\ell'}\delta_{mm'}$, the covariance is diagonal and we obtain

$$\{\mathbf{N}\}_{ii} = \frac{\sigma_0^2}{4\pi}\sum_{\ell=\ell_{\rm min}}^{\ell_{\rm max}}(2\ell+1) = \sigma_0^2\frac{(\ell_{\rm max} - \ell_{\rm min})(\ell_{\rm max} + \ell_{\rm min} + 2)}{4\pi}\,. \tag{345}$$

Therefore, it is possible to construct an approximate noise covariance from simulation if $\langle n^*_{\ell m}n_{\ell'm'}\rangle \sim N_\ell\delta_{\ell\ell'}\delta_{mm'}$ and $N_\ell \sim \sigma_0^2$:

$$\sigma^2(\hat{\boldsymbol{n}}) \equiv \frac{4\pi\{\mathbf{N}\}_{ii}}{(\ell_{\rm max} - \ell_{\rm min})(\ell_{\rm max} + \ell_{\rm min} + 2)}\,. \tag{346}$$

If $N_\ell \not\sim$ const., we need additional operation to the uniform white noise case of Eq. (339):

$$\{\mathbf{Y}^\dagger\mathbf{N}^{-1}\mathbf{Y}\boldsymbol{x}\}_{\ell'm'} = \int \mathrm{d}^2\hat{\boldsymbol{n}}_j\ Y^*_{\ell'm'}(\hat{\boldsymbol{n}}_j)\int \mathrm{d}^2\hat{\boldsymbol{n}}_i\ \left\{H(\hat{\boldsymbol{n}}_i)H(\hat{\boldsymbol{n}}_j)\sum_{LM} Y^*_{LM}(\hat{\boldsymbol{n}}_i)Y_{LM}(\hat{\boldsymbol{n}}_j)N_L^{-1}\right\}\sum_{\ell m} Y_{\ell m}(\hat{\boldsymbol{n}}_i)x_{\ell m}$$
$$= \int \mathrm{d}^2\hat{\boldsymbol{n}}_j\ Y^*_{\ell'm'}(\hat{\boldsymbol{n}}_j)H(\hat{\boldsymbol{n}}_j)\sum_{LM} Y_{LM}(\hat{\boldsymbol{n}}_j)N_L^{-1}\int \mathrm{d}^2\hat{\boldsymbol{n}}_i\ Y^*_{LM}(\hat{\boldsymbol{n}}_i)H(\hat{\boldsymbol{n}}_i)\sum_{\ell m} Y_{\ell m}(\hat{\boldsymbol{n}}_i)x_{\ell m}\,. \tag{347}$$

Here, H is e.g. proportional to square root of a hit count map or $1/W$.

9.3 Preconditioner for the Conjugate Gradient Decent Algorithm

To solve the above equation, we use the preconditioned conjugate gradient decent algorithm. An appropriate preconditioner is essential to solve the equation efficiently. A simple way is to choose the following diagonal preconditioner:

$$\{\mathbf{M}\}_{(\ell m),(\ell m)} = 1 + \sum_k \frac{(b_\ell^k)^2 C_\ell}{2\ell+1} \sum_m \int \mathrm{d}^2\hat{\boldsymbol{n}}_j \; Y^*_{\ell m}(\hat{\boldsymbol{n}}_j) \int \mathrm{d}^2\hat{\boldsymbol{n}}_i \; N_k^{-1}(\hat{\boldsymbol{n}}_i, \hat{\boldsymbol{n}}_j) Y_{\ell m}(\hat{\boldsymbol{n}}_i)\,. \tag{348}$$

For a diagonal noise covariance,

$$\begin{aligned}\{\mathbf{M}\}_{(\ell m),(\ell m)} &= 1 + \sum_k \frac{(b_\ell^k)^2 C_\ell}{2\ell+1} \int \mathrm{d}^2\hat{\boldsymbol{n}}_i \; N_k^{-1}(\hat{\boldsymbol{n}}_i) \sum_m Y^*_{\ell m}(\hat{\boldsymbol{n}}_i) Y_{\ell m}(\hat{\boldsymbol{n}}_i) \\ &= 1 + \sum_k \frac{(b_\ell^k)^2 C_\ell}{4\pi} \int \mathrm{d}^2\hat{\boldsymbol{n}}_i \; N_k^{-1}(\hat{\boldsymbol{n}}_i)\,. \end{aligned} \tag{349}$$

For a given σ_k in unit of μK' and a hit count map, $H_k^2(\hat{\boldsymbol{n}}_i)$, we obtain

$$N_k^{-1}(\hat{\boldsymbol{n}}_i) = H_k^2(\hat{\boldsymbol{n}}_i) \left[\frac{\sigma_k}{T_{CMB}} \times \frac{\pi}{10800}\right]^{-2} \tag{350}$$

Then, we find

$$\{\mathbf{M}\}_{(\ell m),(\ell m)} = 1 + \sum_k \frac{(b_\ell^k)^2 C_\ell}{4\pi} \left[\frac{\sigma_k}{T_{CMB}} \times \frac{\pi}{10800}\right]^{-2} \sum_i \frac{4\pi}{N_{\rm pix}} H_k^2(\hat{\boldsymbol{n}}_i)\,. \tag{351}$$

For a non-uniform noise,

$$\{\mathbf{M}\}_{(\ell m),(\ell m)} = 1 + \sum_k \frac{(b_\ell^k)^2 C_\ell}{4\pi} \int \mathrm{d}^2\hat{\boldsymbol{n}}_j \int \mathrm{d}^2\hat{\boldsymbol{n}}_i \; H_k(\hat{\boldsymbol{n}}_i) H_k(\hat{\boldsymbol{n}}_j) \sum_L N_{L,k}^{-1} P_\ell(\hat{\boldsymbol{n}}_i \cdot \hat{\boldsymbol{n}}_j) \frac{2L+1}{4\pi} P_L(\hat{\boldsymbol{n}}_i \cdot \hat{\boldsymbol{n}}_j)\,. \tag{352}$$

If the noise is close to uniform,

$$\{\mathbf{M}\}_{(\ell m),(\ell m)} \simeq 1 + \sum_k \frac{(b_\ell^k)^2 C_\ell}{4\pi} \langle N_{L,k}^{-1} \rangle_L \int \mathrm{d}^2\hat{\boldsymbol{n}}_j \; H_k^2(\hat{\boldsymbol{n}}_i)\,, \tag{353}$$

where $\langle N_{L,k}^{-1} \rangle_L$ is a representative value of the noise spectrum.

Another way is to split the preconditioner at some scale, $\ell = \ell_{\rm sp}$ and use different preconditioner to these scales. This is motivated by the fact that, for a low resolution map, or if enough computational memory is available, the dense inverse matrix up to $\ell_{\rm sp}$ can be saved. In this case, for the lower multipole, $\ell \leq \ell_{\rm sp}$, the dense inverse matrix is used for the preconditioner while the above approximate diagonal matrix is used for the preconditioner.

This approach is further extended to the multigrid preconditioner. In the multigrid method, we compute the preconditioner at $\ell \leq \ell_{\rm sp}$ from a lower resolution map, while the preconditioner at $\ell > \ell_{\rm sp}$ is given by the above diagonal matrix. At the lower resolution map, the preconditioner is obtained in the same way. By repeating this procedure, at the coarsest map, the preconditioner at $\ell \leq \ell_{\rm sp}$ is obtained by inverting the exact dense matrix.

The dense preconditioning matrix is obtained by substituting $a_{\ell m} = \delta_{\ell\ell_0}\delta_{mm_0}$ for $0 \leq \ell_0 \leq \ell_{\rm sp}$ and $0 \leq m_0 \leq \ell_0$ to the function:

$$a'_{\ell m} = \sum_{\ell' m'} \mathbf{A}_{\ell m, \ell' m'} a_{\ell' m'}\,. \tag{354}$$

Note that the spherical harmonic transform code allows $m \geq 0$ and the above operation gives:

$$a'_{\ell m} = a_{\ell_0 m_0} + \int \mathrm{d}^2\hat{\boldsymbol{n}} \; Y^*_{\ell m}(Y_{\ell_0 m_0} + Y^*_{\ell_0 m_0}) N^{-1}\,. \tag{355}$$

We also substitute $a_{\ell m} = \mathrm{i}\delta_{\ell\ell_0}\delta_{mm_0}$ to obtain

$$a''_{\ell m} = a_{\ell_0 m_0} + \mathrm{i}\int \mathrm{d}^2\hat{\boldsymbol{n}}\; Y^*_{\ell m}(Y_{\ell_0 m_0} - Y^*_{\ell_0 m_0})N^{-1}\,. \tag{356}$$

Then, we obtain the matrix element as

$$\mathbf{A}_{\ell m,\ell_0 m_0} = \frac{a'_{\ell m} - \mathrm{i}a''_{\ell m}}{2}\,. \tag{357}$$

10 Skew-spectrum

10.1 Definition

The skewness relevant to the Minkowski functionals is given by

$$\begin{aligned} S^0(\hat{\boldsymbol{n}}) &\equiv \langle \kappa^3(\hat{\boldsymbol{n}}) \rangle \,, \\ S^1(\hat{\boldsymbol{n}}) &\equiv -3\langle \kappa^2(\hat{\boldsymbol{n}}) \boldsymbol{\nabla}^2 \kappa(\hat{\boldsymbol{n}}) \rangle \,, \\ S^2(\hat{\boldsymbol{n}}) &\equiv -6\langle |\boldsymbol{\nabla} \kappa(\hat{\boldsymbol{n}})|^2 \boldsymbol{\nabla}^2 \kappa(\hat{\boldsymbol{n}}) \rangle \,. \end{aligned} \tag{358}$$

From the above quantities, we obtain

$$\begin{aligned} \overline{S}^0 &= \int d^2\hat{\boldsymbol{n}}\; S^0(\hat{\boldsymbol{n}}) = \int d^2\hat{\boldsymbol{n}} \sum_{\ell_i m_i} Y_{\ell_1 m_1} Y_{\ell_2 m_2} Y_{\ell_3 m_3} \langle \kappa_{\ell_1 m_1} \kappa_{\ell_2 m_2} \kappa_{\ell_3 m_3} \rangle \\ &= \int d^2\hat{\boldsymbol{n}} \sum_{\ell_i m_i} Y_{\ell_1 m_1} Y_{\ell_2 m_2} Y_{\ell_3 m_3} \begin{pmatrix} \ell_1 & \ell_2 & \ell_3 \\ m_1 & m_2 & m_3 \end{pmatrix} h_{\ell_1\ell_2\ell_3} b_{\ell_1\ell_2\ell_3} \\ &= \sum_{\ell_i m_i} \begin{pmatrix} \ell_1 & \ell_2 & \ell_3 \\ m_1 & m_2 & m_3 \end{pmatrix}^2 h^2_{\ell_1\ell_2\ell_3} b_{\ell_1\ell_2\ell_3} \\ &= \sum_{\ell_i} h^2_{\ell_1\ell_2\ell_3} b_{\ell_1\ell_2\ell_3} \,, \end{aligned} \tag{359}$$

$$\begin{aligned} \overline{S}^1 &= \int d^2\hat{\boldsymbol{n}}\; S^1(\hat{\boldsymbol{n}}) = 3\int d^2\hat{\boldsymbol{n}} \sum_{\ell_i m_i} Y_{\ell_1 m_1} Y_{\ell_2 m_2} Y_{\ell_3 m_3} \ell_3(\ell_3+1) \langle \kappa_{\ell_1 m_1} \kappa_{\ell_2 m_2} \kappa_{\ell_3 m_3} \rangle \\ &= 3 \sum_{\ell_i m_i} \ell_3(\ell_3+1) \begin{pmatrix} \ell_1 & \ell_2 & \ell_3 \\ m_1 & m_2 & m_3 \end{pmatrix}^2 h^2_{\ell_1\ell_2\ell_3} b_{\ell_1\ell_2\ell_3} \\ &= \sum_{\ell_i} [\ell_1(\ell_1+1) + \ell_2(\ell_2+1) + \ell_3(\ell_3+1)] h^2_{\ell_1\ell_2\ell_3} b_{\ell_1\ell_2\ell_3} \,, \end{aligned} \tag{360}$$

$$\begin{aligned} \overline{S}^2 &= \int d^2\hat{\boldsymbol{n}}\; S^2(\hat{\boldsymbol{n}}) = 6\int d^2\hat{\boldsymbol{n}} \sum_{\ell_i m_i} \boldsymbol{\nabla} Y_{\ell_1 m_1} \boldsymbol{\nabla} Y_{\ell_2 m_2} \boldsymbol{\nabla}^2 Y_{\ell_3 m_3} \ell_3(\ell_3+1) \langle \kappa_{\ell_1 m_1} \kappa_{\ell_2 m_2} \kappa_{\ell_3 m_3} \rangle \\ &= 3 \sum_{\ell_i m_i} \ell_3(\ell_3+1) [\ell_1(\ell_1+1) + \ell_2(\ell_2+1) - \ell_3(\ell_3+1)] \begin{pmatrix} \ell_1 & \ell_2 & \ell_3 \\ m_1 & m_2 & m_3 \end{pmatrix}^2 h^2_{\ell_1\ell_2\ell_3} b_{\ell_1\ell_2\ell_3} \\ &= \sum_{\ell_i} \{\ell_3(\ell_3+1)[\ell_1(\ell_1+1) + \ell_2(\ell_2+1) - \ell_3(\ell_3+1)] + \text{cyc. perm.}\} h^2_{\ell_1\ell_2\ell_3} b_{\ell_1\ell_2\ell_3} \,. \end{aligned} \tag{361}$$

Here, we use

$$\begin{aligned} I &\equiv \int d^2\hat{\boldsymbol{n}}\; \boldsymbol{\nabla} Y_{\ell_1 m_1} \boldsymbol{\nabla} Y_{\ell_2 m_2} Y_{\ell_3 m_3} \\ &= \ell_2(\ell_2+1) \int d^2\hat{\boldsymbol{n}}\; Y_{\ell_1 m_1} Y_{\ell_2 m_2} Y_{\ell_3 m_3} - \int d^2\hat{\boldsymbol{n}}\; Y_{\ell_1 m_1} \boldsymbol{\nabla} Y_{\ell_2 m_2} \boldsymbol{\nabla} Y_{\ell_3 m_3} \\ &= [\ell_2(\ell_2+1) - \ell_3(\ell_3+1)] \int d^2\hat{\boldsymbol{n}}\; Y_{\ell_1 m_1} Y_{\ell_2 m_2} Y_{\ell_3 m_3} + \int d^2\hat{\boldsymbol{n}}\; \boldsymbol{\nabla} Y_{\ell_1 m_1} Y_{\ell_2 m_2} \boldsymbol{\nabla} Y_{\ell_3 m_3} \\ &= [\ell_2(\ell_2+1) - \ell_3(\ell_3+1) + \ell_1(\ell_1+1)] \int d^2\hat{\boldsymbol{n}}\; Y_{\ell_1 m_1} Y_{\ell_2 m_2} Y_{\ell_3 m_3} - I \,. \end{aligned} \tag{362}$$

10.2 Spectrum

The skew spectra are defined as

$$S_\ell^{(0)} = \frac{1}{2\ell+1}\sum_m \kappa_{\ell m}(\kappa^2)^*_{\ell m} \tag{363}$$

$$S_\ell^{(1)} = \frac{-3}{2\ell+1}\sum_m (\boldsymbol{\nabla}^2\kappa)_{\ell m}(\kappa^2)^*_{\ell m} \tag{364}$$

$$S_\ell^{(2)} = \frac{-6}{2\ell+1}\sum_m (\boldsymbol{\nabla}\kappa\cdot\boldsymbol{\nabla}\kappa)_{\ell m}(\boldsymbol{\nabla}^2\kappa)^*_{\ell m}\,. \tag{365}$$

The expectation values become

$$\langle S_\ell^{(0)}\rangle = \frac{1}{2\ell+1}\sum_m \int \mathrm{d}^2\hat{\boldsymbol{n}} \sum_{\ell_1 m_1 \ell_2 m_2} Y_{\ell m}(\hat{\boldsymbol{n}})Y_{\ell_1 m_1}(\hat{\boldsymbol{n}})Y_{\ell_2 m_2}(\hat{\boldsymbol{n}})\langle\kappa_{\ell m}\kappa_{\ell_1 m_1}\kappa_{\ell_2 m_2}\rangle \tag{366}$$

$$= \frac{1}{2\ell+1}\sum_{\ell_1\ell_2} h^2_{\ell\ell_1\ell_2} b_{\ell\ell_1\ell_2}\,, \tag{367}$$

$$\langle S_\ell^{(1)}\rangle = \frac{3[\ell(\ell+1)]}{2\ell+1}\sum_m \int \mathrm{d}^2\hat{\boldsymbol{n}} \sum_{\ell_1 m_1 \ell_2 m_2} Y_{\ell m}(\hat{\boldsymbol{n}})Y_{\ell_1 m_1}(\hat{\boldsymbol{n}})Y_{\ell_2 m_2}(\hat{\boldsymbol{n}})\langle\kappa_{\ell m}\kappa_{\ell_1 m_1}\kappa_{\ell_2 m_2}\rangle \tag{368}$$

$$= \frac{3[\ell(\ell+1)]}{2\ell+1}\sum_{\ell_1\ell_2} h^2_{\ell\ell_1\ell_2} b_{\ell\ell_1\ell_2}\,, \tag{369}$$

$$\langle S_\ell^{(2)}\rangle = \frac{6[\ell(\ell+1)]}{2\ell+1}\sum_m \int \mathrm{d}^2\hat{\boldsymbol{n}} \sum_{\ell_1 m_1 \ell_2 m_2} Y_{\ell m}(\hat{\boldsymbol{n}})\boldsymbol{\nabla}Y_{\ell_1 m_1}(\hat{\boldsymbol{n}})\boldsymbol{\nabla}Y_{\ell_2 m_2}(\hat{\boldsymbol{n}})\langle\kappa_{\ell m}\kappa_{\ell_1 m_1}\kappa_{\ell_2 m_2}\rangle \tag{370}$$

$$= \frac{3[\ell(\ell+1)]}{2\ell+1}\sum_{\ell_1\ell_2}[-\ell(\ell+1)+\ell_1(\ell_1+1)+\ell_2(\ell_2+1)] h^2_{\ell\ell_1\ell_2} b_{\ell\ell_1\ell_2}\,. \tag{371}$$

The skew spectra, S_ℓ^i, satisfy

$$\overline{S}^i \equiv \sum_\ell (2\ell+1)S_\ell^i\,. \tag{372}$$

11 Other Useful Functions

11.1 Generating correlated random Gaussian fields

We assume that we have N independent random Gaussian fields whose angular power spectra are unity, i.e.:

$$\langle |u^i_{\ell m}|^2 \rangle = 1\,, \tag{373}$$

with $i = 1, 2, \ldots, N$. We then assume that the correlated random Gaussian fields, $a^i_{\ell m}$, are given by

$$a^i_{\ell m} = \sum_j A_{ij,\ell} u^j_{\ell m}\,, \tag{374}$$

where the coefficients, $A_{ij,\ell}$, is determined as follows. First, we impose the following condition:

$$A_{ij,\ell} = 0 \quad (i < j)\,. \tag{375}$$

This means that the matrix, $\{\mathbf{A}_l\}_{ij} = A_{ij,\ell}$, is a lower triangular matrix for each ℓ. The matrix satisfies:

$$c_{ij,\ell} \equiv \{\mathbf{Cov}_\ell\}_{ij} = \langle a^i_{\ell m} (a^j_{\ell m})^* \rangle = \sum_{kn} \langle A_{ik,\ell} u^k_{\ell m} A_{jn,\ell} (u^n_{\ell m})^* \rangle = \sum_k A_{ik,\ell} A_{jk,\ell} = \mathbf{A}_\ell \mathbf{A}^T_\ell\,. \tag{376}$$

The covariance matrix is given by theory. For each ℓ, we obtain (here we omit the subscript, ℓ):

$$c_{ij} = \sum_k A_{ik} A_{jk} = \begin{cases} A_{ii} A_{ji} - \sum_{k=1}^{i-1} A_{ik} A_{jk} & (i < j) \\ \sum_{k=1}^{i} A^2_{ik} & (i = j) \end{cases}\,. \tag{377}$$

Thus, $\mathbf{A}$ is the solution of the Cholesky decomposition of the covariance matrix. We can obtain the coefficients recursively starting from $j = 1$ with $i \le j$ since the covariance is symmetric. If $j = 1$, $i = 1$ and

$$c_{11} = A^2_{11}\,. \tag{378}$$

This means that

$$A_{11} = \sqrt{c_{11}}\,. \tag{379}$$

If $j = 2$,

$$c_{i2} = \begin{cases} A_{11} A_{21} & (i = 1) \\ A^2_{21} + A^2_{22} & (i = 2) \end{cases}\,. \tag{380}$$

Thus, we obtain:

$$A_{21} = \frac{c_{12}}{A_{11}}\,, \tag{381}$$

$$A_{22} = \sqrt{c_{22} - A^2_{21}}\,. \tag{382}$$

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